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Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

From Level M.

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \quad \cdots ①$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \quad \cdots ②$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \quad \cdots ③$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \quad \cdots ④$$

Using the above formulas, obtain the first order derivative of each of the following functions.

(1) $y = \sin x$

[Sol] Using the formula for the derivative of $f(x)$,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} \qquad \text{(from N171a)}$$

$$= \lim_{h\to 0}\frac{2\cos\left(x+\boxed{\frac{h}{2}}\right)\sin\frac{h}{2}}{h} \qquad \text{(from ② above)}$$

$$= \lim_{h\to 0}\left[\cos\left(x+\boxed{\frac{h}{2}}\right)\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]$$

Since $\displaystyle\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} = \boxed{1}$ (from N151b)

Therefore, $\displaystyle\frac{dy}{dx} = \lim_{h\to 0}\left[\cos\left(x+\frac{h}{2}\right)\right]\cdot 1 = \boxed{\cos x}$

01b

(2) $y = \cos x$

[Sol] $$\frac{dy}{dx} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(x + \frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= -\lim_{h \to 0} \left[\sin\left(x + \frac{h}{2}\right) \cdot \frac{\sin\frac{h}{2}}{\frac{h}{2}} \right]$$

$$= -\sin x$$

(3) $y = \tan x$

(**Hint:** From $y = \tan x = \frac{\sin x}{\cos x}$, use the results of $(\sin x)'$ and $(\cos x)'$.)

[Sol] $y' = (\tan x)'$

$$= \left(\frac{\sin x}{\cos x}\right)'$$

$$= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

Formulas $\frac{d}{dx}\sin x = \cos x,\quad \frac{d}{dx}\cos x = -\sin x,\quad \frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$

1. Obtain the first and second order derivatives of each of the following functions.

(1) $y = \sin 5x$

[Sol] Letting $y = \sin z$ and $z = 5x$,

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \cos z\cdot(5x)' = \cos 5x\cdot(5x)' = \boxed{5\cos 5x}$$

Similarly,

$$\frac{d^2y}{dx^2} = 5(-\sin 5x)\cdot(5x)' = \boxed{-25\sin 5x}$$

(2) $y = \cos 2x$

[Sol] $y' = (-\sin 2x)\cdot(2x)' = -2\sin 2x$

$y'' = -2(\sin 2x)' = -2[(\cos 2x)(2x)'] = -4\cos 2x$

(3) $y = \tan 3x$

[Sol] $y' = \left(\frac{1}{\cos^2 3x}\right)\cdot(3x)' = \frac{3}{\cos^2 3x}$

$y'' = 3[(\cos 3x)^{-2}]' = 3[-2(\cos 3x)^{-3}(\cos 3x)']$

$= 3[-2(\cos 3x)^{-3}(-\sin 3x)(3)]$

$= \frac{18\sin 3x}{\cos^3 3x} \quad \left[= \frac{18\tan 3x}{\cos^2 3x}\right]$

0 2 b

2. Obtain the first order derivative of each of the following functions.

(1) $y = \sin^3 x$

[Sol] From $y = (\sin x)^3$, if we let $u = \sin x$ and $y = u^3$, then

$$\frac{dy}{du} = \boxed{3u^2} \qquad \frac{du}{dx} = \boxed{\cos x}$$

Therefore,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot \boxed{\cos x} = \boxed{3\sin^2 x \cos x}$$

Note that the method above is the same as the method below.

$$(u^n)' = nu^{n-1} \cdot u'$$

Thus,

$$y' = 3\sin^2 x(\sin x)' = \boxed{3\sin^2 x \cos x}$$

(2) $y = \cos^2 x$

[Sol] $y' = 2\cos x(\cos x)'$

$= -2\sin x \cos x$

$[= -\sin 2x]$

(3) $y = \tan^2 x$

[Sol] $y' = 2\tan x(\tan x)'$

$= \frac{2\sin x}{\cos^3 x} \quad \left[= \frac{2\tan x}{\cos^2 x}\right]$

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Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	—	1	2	3-

Obtain the first order derivative of each of the following functions.

(1) $y = \sin(3x-2)$

[Sol] $y' = [\cos(3x-2)](3x-2)'$

$= 3\cos(3x-2)$

(2) $y = \cos(3-2x^2)$

[Sol] $y' = [-\sin(3-2x^2)](3-2x^2)'$

$= 4x\sin(3-2x^2)$

(3) $y = \tan\left(\frac{\pi}{3} - \pi x\right)$

[Sol] $y' = \dfrac{1}{\cos^2\left(\frac{\pi}{3} - \pi x\right)} \cdot \left(\frac{\pi}{3} - \pi x\right)'$

$= -\dfrac{\pi}{\cos^2\left(\frac{\pi}{3} - \pi x\right)}$

(4) $y = \sin^2 x$

[Sol] $y' = 2\sin x(\sin x)'$

$= 2\sin x\cos x$

$(= \sin 2x)$

03b

(5) $y = \cos^3 x$

[Sol] $y' = 3\cos^2 x(\cos x)'$

$= 3\cos^2 x(-\sin x)$

$= -3\cos^2 x \sin x$

(6) $y = \tan x^2$

[Sol] $y' = \dfrac{1}{\cos^2 x^2} \cdot (x^2)'$

$= \dfrac{2x}{\cos^2 x^2}$

(2

(7) $y = x\sin x$

[Sol] $y' = x'\sin x + x(\sin x)'$

$= \sin x + x\cos x$

3)

(8) $y = \dfrac{x}{2}\tan x$

[Sol] $y' = \left(\dfrac{x}{2}\right)'\tan x + \dfrac{x}{2}(\tan x)'$

$= \dfrac{1}{2}\tan x + \dfrac{x}{2\cos^2 x}$

Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	1	2	3–

Obtain the first order derivative of each of the following functions.

(1) $y=(2x^2+1)\cos x$

[Sol] $y'=(2x^2+1)'\cos x+(2x^2+1)(\cos x)'$

$=4x\cos x-(2x^2+1)\sin x$

(2) $y=\dfrac{1}{\sin x}$

[Sol] $y'=\left(\dfrac{1}{\sin x}\right)'=-\dfrac{(\sin x)'}{(\sin x)^2}=-\dfrac{\cos x}{\sin^2 x}$ $\left[=-\dfrac{1}{\sin x\tan x}\right]$

(3) $y=\dfrac{1}{\tan 3x}$

[Sol] $y'=\left(\dfrac{1}{\tan 3x}\right)'=-\dfrac{(\tan 3x)'}{(\tan 3x)^2}$

$=-\dfrac{3}{\tan^2 3x\cdot\cos^2 3x}$

$=-\dfrac{3}{\sin^2 3x}$

04b

(4) $y=\dfrac{1}{\cos x}$

[Sol] $y'=\left(\dfrac{1}{\cos x}\right)'=-\dfrac{(\cos x)'}{(\cos x)^2}=-\dfrac{-\sin x}{\cos^2 x}$

$=\dfrac{\sin x}{\cos^2 x}\quad\left[=\dfrac{\tan x}{\cos x}\right]$

(5) $y=\dfrac{1}{\tan x}$

[Sol] $y'=\left(\dfrac{\cos x}{\sin x}\right)'=\dfrac{(\cos x)'\sin x-\cos x(\sin x)'}{(\sin x)^2}$

$=\dfrac{-\sin^2 x-\cos^2 x}{\sin^2 x}=-\dfrac{1}{\sin^2 x}$

(Alternate Solution

$y'=\left(\dfrac{1}{\tan x}\right)'=-\dfrac{(\tan x)'}{(\tan x)^2}$

$=-\dfrac{1}{\tan^2 x\cdot\cos^2 x}=-\dfrac{1}{\sin^2 x}$)

(6) $y=\sin\dfrac{1}{\sqrt{x}}$

[Sol] $y'=\left(\cos\dfrac{1}{\sqrt{x}}\right)\left(\dfrac{1}{\sqrt{x}}\right)'=-\dfrac{1}{2x\sqrt{x}}\cos\dfrac{1}{\sqrt{x}}\qquad\left[=-\dfrac{\cos\dfrac{1}{\sqrt{x}}}{2x\sqrt{x}}\right]$

(7) $y=(\sin x+\cos x)^3$

[Sol] $y'=3(\sin x+\cos x)^2(\sin x+\cos x)'$

$=3(\sin x+\cos x)^2(\cos x-\sin x)$

Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	2	3–

Obtain the first order derivative of each of the following functions.

(1) $y = \sin x(1 + \cos x)$

[Sol] $y' = (\sin x)'(1 + \cos x) + \sin x(1 + \cos x)'$

$= \cos x(1 + \cos x) - \sin^2 x$

$= 2\cos^2 x + \cos x - 1 \quad [= (2\cos x - 1)(\cos x + 1)]$

(2) $y = \sin 2x \cos 3x$

[Sol] $y' = \cos 2x(2x)'\cos 3x + \sin 2x(-\sin 3x)(3x)'$

$= 2\cos 2x \cos 3x - 3\sin 2x \sin 3x$

$= (\cos 5x + \cos x) + \dfrac{3}{2}(\cos 5x - \cos x)$

$= \dfrac{1}{2}(5\cos 5x - \cos x)$

(Alternate Solution

$y = \sin 2x \cos 3x = \dfrac{1}{2}(\sin 5x - \sin x)$

$\therefore y' = \dfrac{1}{2}(5\cos 5x - \cos x)$)

(3) $y = \sin 2x \tan x$

[Sol] $y' = \left(2\sin x \cos x \cdot \dfrac{\sin x}{\cos x}\right)' = 2(\sin^2 x)'$

$= 4\sin x(\sin x)'$

$= 4\sin x \cos x$

$= 2\sin 2x$

05b

(4) $y=\sin^2 x\cos 2x$

[Sol] $y' = 2\sin x(\sin x)'\cos 2x+\sin^2 x(-\sin 2x)(2x)'$

$= 2\sin x\cos x\cos 2x-2\sin^2 x\sin 2x$

$= \sin 2x\cos 2x-2\sin^2 x\sin 2x$

$= \sin 2x(\cos 2x-2\sin^2 x)$

$$\left(\begin{aligned}&\text{Alternate Solution}\\ &y'=\sin 2x\left(\cos 2x-2\cdot\frac{1-\cos 2x}{2}\right)\\ &\quad=\sin 2x(2\cos 2x-1)\\ &\quad=2\sin 2x\cos 2x-\sin 2x\\ &\quad=\sin 4x-\sin 2x\end{aligned}\right)$$

(5) $y=\sin^3 x\cos^2 x$

[Sol] $y' = 3(\sin^2 x)(\sin x)'(\cos^2 x)+(\sin^3 x)\cdot 2\cos x(\cos x)'$

$= 3\sin^2 x\cos^3 x-2\cos x\sin^4 x$

$= \sin^2 x\cos x(3\cos^2 x-2\sin^2 x)$

$= \sin^2 x\cos x(3-5\sin^2 x)$

(6) $y=\sin x\cos^2 x$

[Sol] $y' = (\sin x)'\cos^2 x+\sin x\cdot 2\cos x(\cos x)'$

$= \cos^3 x-2\cos x\sin^2 x$

$= \cos^3 x-2\cos x(1-\cos^2 x)$

$= 3\cos^3 x-2\cos x$

(7) $y=\sin 3x+\cos x^3$

[Sol] $y' = (\sin 3x)'+(\cos x^3)'$

$= 3\cos 3x-\sin x^3\cdot(x^3)'$

$= 3\cos 3x-3x^2\sin x^3$

Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	—	1	2	3-

Obtain the first order derivative of each of the following functions.

(1) $y=\sqrt{\cos 3x}$

[Sol] $y'=\dfrac{1}{2\sqrt{\cos 3x}}(\cos 3x)'$

$=-\dfrac{3\sin 3x}{2\sqrt{\cos 3x}}$

(2) $y=\sqrt{1-\sin x}$

[Sol] $y'=\dfrac{1}{2\sqrt{1-\sin x}}(1-\sin x)'$

$=-\dfrac{\cos x}{2\sqrt{1-\sin x}}$

(3) $y=\dfrac{1}{\sqrt{\tan x}}$

[Sol] $y=(\tan x)^{-\frac{1}{2}}$

$y'=-\dfrac{1}{2}\cdot\dfrac{1}{\tan x\sqrt{\tan x}}\cdot\dfrac{1}{\cos^2 x}$

$=-\dfrac{1}{2\sin x\cos x\sqrt{\tan x}}$

$\left[=-\dfrac{1}{2\sqrt{\sin^3 x\cos x}}\right]$

06b

(4) $y = \sin^2(3x+2)$

[Sol] $y' = 2\sin(3x+2)[\sin(3x+2)]'$

$= 6\sin(3x+2)\cos(3x+2)$

$= 3\sin(6x+4)$

(5) $y = \cos^3(1-2x^2)$

[Sol] $y' = 3\cos^2(1-2x^2)[\cos(1-2x^2)]'$

$= 12x\cos^2(1-2x^2)\sin(1-2x^2)$

(6) $y = \dfrac{\cos x}{x}$

[Sol] $y' = \left(\dfrac{\cos x}{x}\right)' = \dfrac{(\cos x)'\cdot x - \cos x}{x^2}$

$= -\dfrac{x\sin x + \cos x}{x^2}$

(7) $y = \dfrac{\cos x}{1+\cos x}$

[Sol] $y' = \left(\dfrac{\cos x}{1+\cos x}\right)' = \dfrac{(\cos x)'(1+\cos x) - \cos x(1+\cos x)'}{(1+\cos x)^2}$

$= \dfrac{-\sin x - \sin x\cos x + \cos x\sin x}{(1+\cos x)^2}$

$= -\dfrac{\sin x}{(1+\cos x)^2}$

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Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

1. Obtain the first order derivative of the exponential function $y = f(x) = a^x$.

[Sol] Using the formula for the derivative of $f(x)$,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

Since $a^{x+h} = a^{\boxed{x}} \cdot a^{\boxed{h}}$,

$$\frac{dy}{dx} = \lim_{h \to 0} \left(a^x \cdot \boxed{\frac{a^h - 1}{h}} \right) = a^x \lim_{h \to 0} \boxed{\frac{a^h - 1}{h}}$$

$$= a^x \lim_{h \to 0} \frac{a^h - a^{\boxed{0}}}{h}$$

$$= a^x f'(0)$$

The value of $f'(0)$ is the derivative of $y = f(x) = a^x$ at $x = 0$.

The value of $f'(0)$ is also the slope of the tangent line at $x = 0$.

Assuming that e is the value of a at which the slope of the tangent line at $f(0)$ is 1, the derivative becomes:

$$\frac{dy}{dx} = (e^x)' = e^x \cdot 1 = \boxed{e^x}$$

$$\left[e = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = 2.7182818\ldots \right]$$

From the definition of the logarithm,

$$a = e^{\log_e a}$$

Therefore, $a^x = e^{x \log_e a}$ and

$$(a^x)' = (e^{x \log_e a})'$$
$$= e^{x \log_e a} \cdot (x \log_e a)'$$
$$= a^x \log_e a$$

When a is a positive number, the logarithm of a to the base e is called the *natural logarithm*, and is usually written as **ln *a*** (where $\ln a = \log_e a$).

Formulas $\boxed{(e^x)' = e^x, \quad (a^x)' = a^x \ln a}$

2. Obtain the first order derivative of each of the following functions.

Ex. $y = e^{2x}$

[Sol] $y' = e^{2x}(2x)' = 2e^{2x}$

(1) $y = e^x$

[Sol] $y' = e^x (x)' = e^x$

(2) $y = e^{-3x}$

[Sol] $y' = e^{-3x}(-3x)' = -3e^{-3x}$

(3) $y = 2^x$

[Sol] $y' = 2^x \ln 2$

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Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	1	2	3-

1. Obtain the first and second order derivatives of each of the following functions.

(1) $y = e^x$

[Sol] $y' = e^x$

$y'' = e^x$

(2) $y = e^{-x^3}$

[Sol] $y' = e^{-x^3}(-x^3)' = -3x^2e^{-x^3}$

$y'' = -3[(x^2)' \cdot e^{-x^3} + x^2 \cdot (e^{-x^3})']$

$= -3[(2x) \cdot e^{-x^3} + x^2 \cdot (-3x^2e^{-x^3})]$

$= -3xe^{-x^3}(2 - 3x^3)$

$[= 3xe^{-x^3}(3x^3 - 2)]$

(3) $y = e^x + e^{-x}$

[Sol] $y' = e^x(x)' + e^{-x}(-x)'$

$= e^x - e^{-x}$

$y'' = (e^x)' - (e^{-x})'$

$= e^x + e^{-x}$

08b

2. Obtain the first order derivative of each of the following functions.

(1) $y = a^{3x}$

[Sol] $y' = a^{3x} \cdot \ln a \cdot (3x)' = 3a^{3x} \cdot \ln a$

(2) $y = a^{-2x}$

[Sol] $y' = a^{-2x} \cdot \ln a \cdot (-2x)'$

$= -2a^{-2x} \cdot \ln a$

(3) $y = xa^{x}$

[Sol] $y' = a^{x} + xa^{x} \cdot \ln a$

$= a^{x}(1 + x\ln a)$

(4) $y = (e^{x} + e^{-x})^{2}$

[Sol] $y' = 2(e^{x} + e^{-x})(e^{x} + e^{-x})'$

$= 2(e^{x} + e^{-x})(e^{x} - e^{-x})$

$= 2(e^{2x} - e^{-2x})$

Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	1	2	3–

Obtain the first order derivative of each of the following functions.

(1) $y=(x-x^2)e^x$

[Sol] $y'=(x-x^2)'e^x+(x-x^2)(e^x)'$

$=(1-2x)e^x+(x-x^2)e^x$

$=(1-x-x^2)e^x$

(2) $y=xe^{-x^2}$

[Sol] $y'=(x)'e^{-x^2}+x(e^{-x^2})'$

$=e^{-x^2}+xe^{-x^2}(-x^2)'$

$=e^{-x^2}-2x^2e^{-x^2}$

$=(1-2x^2)e^{-x^2}$

(3) $y=e^{2x}\sin 3x$

[Sol] $y'=2e^{2x}\sin 3x+e^{2x}\cdot 3\cos 3x$

$=e^{2x}(2\sin 3x+3\cos 3x)$

(4) $y=\dfrac{e^x}{x}$

[Sol] $y'=\dfrac{e^x\cdot x-e^x\cdot 1}{x^2}=\dfrac{(x-1)e^x}{x^2}$

09b

(5) $y=\dfrac{e^x-1}{e^x+1}$

[Sol] $y'=\dfrac{e^x(e^x+1)-(e^x-1)e^x}{(e^x+1)^2}$

$=\dfrac{2e^x}{(e^x+1)^2}$

(6) $y=\dfrac{\sqrt{x}}{e^x}$

[Sol] $y'=\dfrac{\dfrac{1}{2\sqrt{x}}\cdot e^x-\sqrt{x}\cdot e^x}{e^{2x}}$

$=\dfrac{e^x-2xe^x}{2\sqrt{x}\cdot e^{2x}}$

$=\dfrac{1-2x}{2e^x\sqrt{x}}$

(7) $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

[Sol] $y'=\dfrac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$

$=\dfrac{4e^x\cdot e^{-x}}{(e^x+e^{-x})^2}$

$=\dfrac{4}{(e^x+e^{-x})^2}$

Alternate Solution

$y=\dfrac{e^{2x}-1}{e^{2x}+1}$

$y'=\dfrac{2e^{2x}(e^{2x}+1)-(e^{2x}-1)(2e^{2x})}{(e^{2x}+1)^2}$

$=\dfrac{4e^{2x}}{(e^{2x}+1)^2}$

Advanced Differentiation 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

1. Obtain the first and second order derivatives of each of the following functions.

(1) $y = xe^x$

[Sol] $y' = x'e^x + x(e^x)'$

$= e^x + xe^x = e^x(1+x)$

$y'' = e^x + e^x(1+x)$

$= e^x(2+x)$

(2) $y = -\cos 2x + \sin 3x$

[Sol] $y' = (\sin 2x)(2x)' + (\cos 3x)(3x)'$

$= 2\sin 2x + 3\cos 3x$

$y'' = 2\cos 2x(2x)' - 3\sin 3x(3x)'$

$= 4\cos 2x - 9\sin 3x$

2. Obtain the first order derivative of each of the following functions.

(1) $y = \dfrac{1}{\tan 2x}$

[Sol] $y' = \left(\dfrac{1}{\tan 2x}\right)' = -\dfrac{(\tan 2x)'}{(\tan 2x)^2}$

$= -\dfrac{2}{\tan^2 2x \cdot \cos^2 2x}$

$= -\dfrac{2}{\sin^2 2x}$

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(2) $y = \sin 4x \cos 3x$

[Sol] $y' = \cos 4x(4x)' \cos 3x + \sin 4x(-\sin 3x)(3x)'$

$= 4\cos 4x \cos 3x - 3\sin 4x \sin 3x$

$= 2(\cos 7x + \cos x) + \frac{3}{2}(\cos 7x - \cos x)$

$= \frac{1}{2}(7\cos 7x + \cos x)$

(3) $y = \sqrt{1-\cos x}$

[Sol] $y' = \frac{1}{2\sqrt{1-\cos x}}(1-\cos x)'$

$= \frac{\sin x}{2\sqrt{1-\cos x}}$

(4) $y = (e^x + e^{-x})^3$

[Sol] $y' = 3(e^x + e^{-x})^2(e^x + e^{-x})'$

$= 3(e^x + e^{-x})^2(e^x - e^{-x})$

Advanced Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

1. Differentiate the logarithmic function $y = f(x) = \ln x$. $(x > 0)$

[Sol] Using the formula for the derivative of $f(x)$,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$= \lim_{h \to 0}\left[\frac{1}{h}\left(\ln\frac{x+h}{x}\right)\right]$$

$$= \lim_{h \to 0}\left[\frac{1}{h}\ln\left(1+\frac{h}{x}\right)\right]$$

Substituting k for $\frac{h}{x}$, as $h \to 0$ then $k \to 0$.

$$\frac{dy}{dx} = \lim_{k \to 0}\left[\frac{1}{kx}\ln(1+k)\right]$$

$$= \lim_{k \to 0}\left[\frac{1}{x}\ln(1+k)^{\frac{1}{k}}\right]$$

However, since $\lim_{t \to \infty}\left(1+\frac{1}{t}\right)^t = e$,

$$\lim_{k \to 0}\ln(1+k)^{\frac{1}{k}} = \boxed{1}$$

$$\therefore \frac{dy}{dx} = \lim_{k \to 0}\left[\frac{1}{x}\ln(1+k)^{\frac{1}{k}}\right]$$

$$= \frac{1}{x}\cdot\boxed{1} = \boxed{\frac{1}{x}}$$

[Using the formula for differentiation of composite functions when $x < 0$,

$$[\ln(-x)]' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$$
]

Formula $\boxed{(\ln|x|)' = \frac{1}{x}}$

O 11 b

2. Differentiate the function $y = \log_a x$ using the result of side a, where $a > 0$, $a \neq 1$, and $x > 0$.

(For base conversion, use the formula $\boxed{\log_a b = \frac{\ln b}{\ln a}}$.)

[Sol] $y = \log_a x = \frac{\ln x}{\ln a}$

$y' = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$

3. Differentiate each of the following functions.

(1) $y = \ln(3x - 1)$

[Sol] $y' = \frac{1}{3x - 1} \cdot (3x - 1)'$

$= \frac{3}{3x - 1}$

(2) $y = (\ln x)^2$

[Sol] $y' = 2 \ln x \cdot (\ln x)' = \frac{2 \ln x}{x}$

KUMON

O 12 a

Advanced Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	2	3–

Differentiate each of the following functions.

(1) $y = \log_2(1-2x)$

[Sol] $y = \dfrac{\ln(1-2x)}{\ln 2}$

$$y' = \frac{\dfrac{-2}{1-2x}}{\ln 2} = \frac{2}{(2x-1)\ln 2}$$

(2) $y = \ln(\cos x)$

[Sol] $y' = \dfrac{-\sin x}{\cos x} = -\tan x$

(3) $y = \ln(x^2 - x + 1)$

[Sol] $y' = \dfrac{2x-1}{x^2-x+1}$

(4) $y = \dfrac{\ln x}{x}$

[Sol] $y' = \dfrac{\dfrac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \dfrac{1-\ln x}{x^2}$

O 12 b

Ex. $y = \log_4 x$

[Sol] $y = \dfrac{\ln x}{\ln 4}$

$$y' = \frac{1}{\ln 4}(\ln x)' = \frac{1}{x \ln 4}$$

(5) $y = \log_{10}(3x + 1)$

[Sol] $y = \dfrac{\ln(3x + 1)}{\ln 10}$

$$y' = \frac{1}{\ln 10} \cdot [\ln(3x + 1)]' = \frac{3}{(3x + 1)\ln 10}$$

(6) $y = \dfrac{1}{x}\ln x^2$

[Sol] $y' = -\dfrac{1}{x^2}\ln x^2 + \dfrac{1}{x} \cdot \dfrac{1}{x^2} \cdot 2x$

$$= \frac{2 - \ln x^2}{x^2}$$

(7) $y = x^2 \log_3 x$

[Sol] $y = x^2 \dfrac{\ln x}{\ln 3}$

$$y' = \frac{1}{\ln 3}(x^2 \ln x)'$$

$$= \frac{1}{\ln 3}\left(2x \cdot \ln x + x^2 \cdot \frac{1}{x}\right)$$

$$= \frac{x(2\ln x + 1)}{\ln 3}$$

KUMON

O 13 a

Advanced Differentiation 2

Time   :   to   :   Date   Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

Differentiate each of the following functions.

(1) $y=x^2(\ln x)^3$

[Sol] $y'=2x\cdot(\ln x)^3+x^2\cdot 3(\ln x)^2\dfrac{1}{x}$

$=x(\ln x)^2(2\ln x+3)$

(2) $y=\ln(x+\sqrt{x^2+1})$

[Sol] $y'=\dfrac{(x+\sqrt{x^2+1})'}{x+\sqrt{x^2+1}}$

$=\dfrac{1+\dfrac{2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$

$=\dfrac{1}{\sqrt{x^2+1}}$

$\left[=\dfrac{\sqrt{x^2+1}}{x^2+1}\right]$

(3) $y=(\sqrt{2x+1})^3\ln 2x$

[Sol] $y=(2x+1)^{\frac{3}{2}}\ln 2x$

$y'=\dfrac{3}{2}(2x+1)^{\frac{1}{2}}(2)\cdot\ln 2x+(2x+1)^{\frac{3}{2}}\cdot\dfrac{1}{2x}(2)$

$=3\ln 2x\sqrt{2x+1}+\dfrac{(2x+1)\sqrt{2x+1}}{x}$

$\left[=\dfrac{\sqrt{2x+1}(3x\ln 2x+2x+1)}{x}\right]$

O 13 b

(4) $y = \ln\dfrac{1-\sin x}{1+\sin x}$

[Sol] $y = \ln(1-\sin x) - \ln(1+\sin x)$

$$y' = [\ln(1-\sin x)]' - [\ln(1+\sin x)]'$$

$$= \frac{-\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x}$$

$$= \cos x\left(\frac{1}{\sin x - 1} - \frac{1}{\sin x + 1}\right)$$

$$= \frac{2\cos x}{\sin^2 x - 1}$$

$$= -\frac{2}{\cos x}$$

(5) $y = \ln\dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x}$

[Sol] $y = \ln(\sqrt{x^2+1}+x) - \ln(\sqrt{x^2+1}-x)$

$$y' = [\ln(\sqrt{x^2+1}+x)]' - [\ln(\sqrt{x^2+1}-x)]'$$

$$= \frac{\frac{2x}{2\sqrt{x^2+1}}+1}{\sqrt{x^2+1}+x} - \frac{\frac{2x}{2\sqrt{x^2+1}}-1}{\sqrt{x^2+1}-x}$$

$$= \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}(\sqrt{x^2+1}+x)} - \frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}(\sqrt{x^2+1}-x)}$$

$$= \frac{1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{2}{\sqrt{x^2+1}}$$

$$\left[= \frac{2\sqrt{x^2+1}}{x^2+1}\right]$$

KUMON
O 14 a

Advanced Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

1. Differentiate the following function.

$$y = x^x \qquad (x > 0)$$

[Sol] This function cannot be differentiated as it is.

Taking the natural logarithm of both sides,

$$\ln y = x \ln x$$

Differentiating both sides with respect to x,

$$\text{LHS} = \frac{d}{dx}\ln y = \frac{d}{dy}\ln y \cdot \frac{dy}{dx} = \boxed{\frac{1}{y}} \cdot \frac{dy}{dx}$$

$$\text{RHS} = \boxed{\ln x + 1}$$

Therefore,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \boxed{\ln x + 1}$$

$$\therefore \frac{dy}{dx} = \boxed{x^x(\ln x + 1)}$$

This method is called *logarithmic differentiation.*

O 14 b

2. Differentiate each of the following functions.

(1) $y=7^x$

[Sol] Taking the natural logarithm of both sides,

$$\ln y = x\ln 7$$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\ln 7$$

$$\therefore\ y'=7^x\ln 7$$

(2) $y=x^{2x}$ $(x>0)$

[Sol] Taking the natural logarithm of both sides,

$$\ln y = 2x\ln x$$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=2\ln x+2$$

$$\therefore\ y'=2x^{2x}(\ln x+1)$$

(3) $y=2^x\sin x$

[Sol] Taking the natural logarithm of both sides,

$$\ln|y| = x\ln 2+\ln|\sin x|$$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\ln 2+\frac{\cos x}{\sin x}$$

$$\therefore\ y'=2^x(\ln 2\cdot\sin x+\cos x)$$

Advanced Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Differentiate each of the following functions.

(1) $y = x^5(3x-1)^3$

[Sol] Rewriting, $\boxed{|y|} = |x|^5 \cdot |\boxed{3x-1}|^3$

Taking the natural logarithm of both sides,

$$\ln|y| = 5\ln|x| + 3\ln|3x-1|$$

Differentiating both sides with respect to x,

$$\frac{y'}{y} = \frac{5}{x} + 3\cdot\frac{3}{3x-1} = \frac{24x-5}{x(3x-1)}$$

$$\therefore\ y' = x^4(3x-1)^2(24x-5)$$

(2) $y = \dfrac{(x+1)^3}{(x-2)^2(x+3)^4}$

[Sol] Rewriting, $|y| = \dfrac{|x+1|^3}{|x-2|^2\cdot|x+3|^4}$

Taking the natural logarithm of both sides,

$$\ln|y| = 3\ln|x+1| - 2\ln|x-2| - 4\ln|x+3|$$

Differentiating both sides with respect to x,

$$\frac{y'}{y} = \frac{3}{x+1} - \frac{2}{x-2} - \frac{4}{x+3}$$

$$= -\frac{3x^2+x+16}{(x+1)(x-2)(x+3)}$$

$$\therefore\ y' = -\frac{3x^2+x+16}{(x+1)(x-2)(x+3)}\cdot\frac{(x+1)^3}{(x-2)^2(x+3)^4}$$

$$= -\frac{(x+1)^2(3x^2+x+16)}{(x-2)^3(x+3)^5}$$

O 15 b

(3) $y=\sqrt{\dfrac{(x-3)^3}{(x-1)(x-2)}}$

[Sol] Rewriting, $y=\left[\dfrac{(x-3)^3}{(x-1)(x-2)}\right]^{\frac{1}{2}}$

Taking the natural logarithm of both sides,

$$\ln y=\frac{1}{2}(3\ln|x-3|-\ln|x-1|-\ln|x-2|)$$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\frac{1}{2}\left(\frac{3}{x-3}-\frac{1}{x-1}-\frac{1}{x-2}\right)$$

$$=\frac{x^2-3}{2(x-1)(x-2)(x-3)}$$

$$\therefore y'=\frac{x^2-3}{2(x-1)(x-2)(x-3)}\cdot\sqrt{\frac{(x-3)^3}{(x-1)(x-2)}}$$

$$=\frac{(x^2-3)\sqrt{(x-1)(x-2)(x-3)}}{2(x-1)^2(x-2)^2}$$

(4) $y=x^{\ln x}$ $(x>0)$

[Sol] Taking the natural logarithm of both sides,

$$\ln y=(\ln x)^2$$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=2\ln x\cdot\frac{1}{x}=\frac{2\ln x}{x}$$

$$\therefore y'=\frac{2\ln x}{x}\cdot x^{\ln x}$$

$$=2x^{\ln x-1}\cdot\ln x$$

Advanced Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2–

In each exercise, obtain the value of $\frac{dy}{dx}$ of the given function. Express the answers in terms of x and y.

(1) $ax^2 + by^2 = 1 \qquad (b \neq 0) \qquad \ldots①$

[Sol 1] Solving ① for y,

$$y = \pm\sqrt{\frac{1-ax^2}{b}}$$

(a) When $y = \sqrt{\frac{1-ax^2}{b}}$,

$$y' = \frac{\frac{-2ax}{b}}{2\sqrt{\frac{1-ax^2}{b}}} = \frac{-ax}{b\sqrt{\frac{1-ax^2}{b}}} = -\frac{ax}{\boxed{by}}$$

(b) When $y = -\sqrt{\frac{1-ax^2}{b}}$,

$$y' = \frac{ax}{b\sqrt{\frac{1-ax^2}{b}}} = -\frac{ax}{by}$$

$\therefore$ From (a) and (b),

$$\frac{dy}{dx} = -\frac{ax}{by}$$

[Sol 2] Differentiating both sides of ① with respect to x,

$$2ax + 2by \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{ax}{by}$$

O 16 b

(2) $x^2+y^2=r^2$

[Sol] Differentiating both sides with respect to x,

$$2x+2y\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{x}{y}$$

(3) $4x^2-9y^2=36$

[Sol] Differentiating both sides with respect to x,

$$8x-18y\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{4x}{9y}$$

(4)

(4) $x^2+2xy-5y^2=1$

[Sol] Differentiating both sides with respect to x,

$$2x+2\left(y+x\frac{dy}{dx}\right)-10y\frac{dy}{dx}=0$$

Therefore, $(5y-x)\frac{dy}{dx}=x+y$

$$\therefore \frac{dy}{dx}=\frac{x+y}{-x+5y}$$

O 17 a Advanced Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

In each exercise, obtain the value of $\dfrac{dy}{dx}$ of the given function.

(1) $\sqrt{x}+\sqrt{y}=\sqrt{a}$

[Sol] Differentiating both sides with respect to x,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

(2) $(y+1)^2=x^2-x$

[Sol] Differentiating both sides with respect to x,

$$2(y+1)\frac{dy}{dx}=2x-1$$

$$\therefore \frac{dy}{dx}=\frac{2x-1}{2(y+1)}$$

(3) $xy=1$

[Sol] Differentiating both sides with respect to x,

$$y+x\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{y}{x}$$

O 17 b

(4) $y^2 = 4px$

[Sol] Differentiating both sides with respect to x,

$$2y\frac{dy}{dx} = 4p$$

$$\therefore \frac{dy}{dx} = \frac{2p}{y}$$

(5) $\sin x + \sin y = 1$

[Sol] Differentiating both sides with respect to x,

$$\cos x + \cos y \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\cos x}{\cos y}$$

(6) $\ln(x+y) = x$

[Sol] Differentiating both sides with respect to x,

$$\frac{1 + \dfrac{dy}{dx}}{x+y} = 1$$

$$\therefore \frac{dy}{dx} = x + y - 1$$

O 18 a Advanced Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

In each exercise, using the given equations, obtain the value of $\dfrac{dy}{dx}$.

(1)
$$f(x)\begin{cases} x = t + \dfrac{1}{t} & \dots ① \\ y = t - \dfrac{1}{t} & \dots ② \end{cases}$$

[Sol 1] From ①^2 − ②^2,

$$x^2 - y^2 = 4$$

Differentiating both sides with respect to x,

$$2x - 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{x}{y} = \frac{t^2+1}{t^2-1}$$

[Sol 2] Since $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$,

$$\frac{dx}{dt} = 1 - \frac{1}{t^2}, \quad \frac{dy}{dt} = 1 + \frac{1}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2+1}{t^2-1}$$

When $x = f(t)$ and $y = g(t)$, $\quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \left(\text{where } \dfrac{dx}{dt} \neq 0\right)$

O 18 b

(2) $x=t+1,\quad y=3t+2$

[Sol] $\dfrac{dx}{dt}=1,\quad \dfrac{dy}{dt}=3$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=3$$

(3) $x=t+\dfrac{1}{t},\quad y=t^2+\dfrac{1}{t^2}$

[Sol] $\dfrac{dx}{dt}=1-\dfrac{1}{t^2},\quad \dfrac{dy}{dt}=2t-\dfrac{2}{t^3}$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2(t^2+1)}{t}$$

(4) $x=\dfrac{2t}{1+t^2},\quad y=\dfrac{1-t^2}{1+t^2}$

[Sol] $\dfrac{dx}{dt}=\dfrac{2(1+t^2)-2t\cdot 2t}{(1+t^2)^2}=\dfrac{2(1-t^2)}{(1+t^2)^2}$

$\dfrac{dy}{dt}=\dfrac{-2t(1+t^2)-(1-t^2)\cdot 2t}{(1+t^2)^2}=-\dfrac{4t}{(1+t^2)^2}$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=-\frac{2t}{1-t^2}$$

KUMON

O 19 a Advanced Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	1	2-

In each exercise, using the given equations, obtain the value of $\frac{dy}{dx}$.

(1) $x = \sin t, \quad y = \cos t$

[Sol] $\frac{dx}{dt} = \cos t$

$\frac{dy}{dt} = -\sin t$

$\therefore \frac{dy}{dx} = -\frac{\sin t}{\cos t} = -\tan t$

(2) $x = a\sin^3 t, \quad y = b\cos^3 t$

[Sol] $\frac{dx}{dt} = 3a\sin^2 t\cos t$

$\frac{dy}{dt} = 3b\cos^2 t(-\sin t)$

$\therefore \frac{dy}{dx} = \frac{-3b\cos^2 t\sin t}{3a\sin^2 t\cos t} = -\frac{b}{a\tan t}$

O 19 b

(3) $x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t)$

[Sol] $$\frac{dx}{dt} = a[(-\sin t) + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a[\cos t - (\cos t - t\sin t)] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{at\sin t}{at\cos t} = \tan t$$

(4) $x = \dfrac{\sin t}{t + \tan t}, \quad y = \dfrac{\cos t}{t + \tan t}$

[Sol] $$\frac{dx}{dt} = \frac{(\cos t)(t + \tan t) - (\sin t)\left(1 + \dfrac{1}{\cos^2 t}\right)}{(t + \tan t)^2}$$

$$= \frac{(\cos t)\left(\dfrac{t\cos t + \sin t}{\cos t}\right) - (\sin t)\left(\dfrac{\cos^2 t + 1}{\cos^2 t}\right)}{\left(\dfrac{t\cos t + \sin t}{\cos t}\right)^2}$$

$$= \frac{t\cos^3 t - \sin t}{(t\cos t + \sin t)^2}$$

$$\frac{dy}{dt} = \frac{(-\sin t)(t + \tan t) - (\cos t)\left(1 + \dfrac{1}{\cos^2 t}\right)}{(t + \tan t)^2}$$

$$= \frac{-t\sin t\cos^2 t - \cos t(\sin^2 t + \cos^2 t) - \cos t}{(t\cos t + \sin t)^2}$$

$$= \frac{-t\sin t\cos^2 t - 2\cos t}{(t\cos t + \sin t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{-t\sin t\cos^2 t - 2\cos t}{t\cos^3 t - \sin t}$$

Advanced Differentiation 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	1	2~

1. Obtain the value of $\frac{dy}{dx}$ of the following function.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where a and b are positive constants)

[Sol] Differentiating both sides with respect to x,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

Therefore, $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$

2. Using the given equations, obtain the value of $\frac{dy}{dx}$.

$x=2t+\frac{1}{t^2}$, $y=t-\frac{1}{t^3}$

[Sol] $\frac{dx}{dt}=2-\frac{2}{t^3}$, $\frac{dy}{dt}=1+\frac{3}{t^4}$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1+\frac{3}{t^4}}{2-\frac{2}{t^3}}=\frac{t^4+3}{2t(t^3-1)} \qquad \left[=\frac{t^4+3}{2t(t-1)(t^2+t+1)}\right]$$

3. Differentiate each of the following functions.

(1) $y=\sqrt{\frac{1+\sin x}{1-\sin x}}$

[Sol] Taking the natural logarithm of both sides,

$$\ln y=\frac{1}{2}[\ln(1+\sin x)-\ln(1-\sin x)]$$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\frac{1}{2}\left(\frac{\cos x}{1+\sin x}-\frac{-\cos x}{1-\sin x}\right)=\frac{1}{2}\cdot\frac{2\cos x}{1-\sin^2 x}$$

$$=\frac{1}{\cos x}$$

Therefore, $y'=\frac{1}{\cos x}\sqrt{\frac{1+\sin x}{1-\sin x}}$

O 20 b

(2) $y = \sqrt{\dfrac{(x-2)^2}{(x-3)(x-1)}}$

[Sol] Rewriting.

$$y = \left[\frac{(x-2)^2}{(x-3)(x-1)}\right]^{1}$$

Taking the natural logarithm of both sides,

$$\ln y = \frac{1}{2}(2\ln|x-2| - \ln|x-3| - \ln|x-1|)$$

Differentiating both sides with respect to x,

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{2}{x-2} - \frac{1}{x-3} - \frac{1}{x-1}\right)$$

$$= -\frac{1}{(x-1)(x-2)(x-3)}$$

$$\therefore y' = -\frac{1}{(x-1)(x-2)(x-3)} \cdot \sqrt{\frac{(x-2)^2}{(x-3)(x-1)}}$$

$$= -\frac{\sqrt{(x-1)(x-3)}}{(x-1)^2(x-3)^2}$$

Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Given the fractional function $y = \dfrac{x^2}{x+1}$,

$\lim\limits_{x \to -1^+} y = +\infty$, $\lim\limits_{x \to -1^-} y = -\infty$, and $\lim\limits_{x \to \pm\infty} [y - (x-1)] = \lim\limits_{x \to \pm\infty} \dfrac{1}{x+1} = 0$

Therefore, the equations of the asymptotes are $x = -1$, $y = x - 1$.

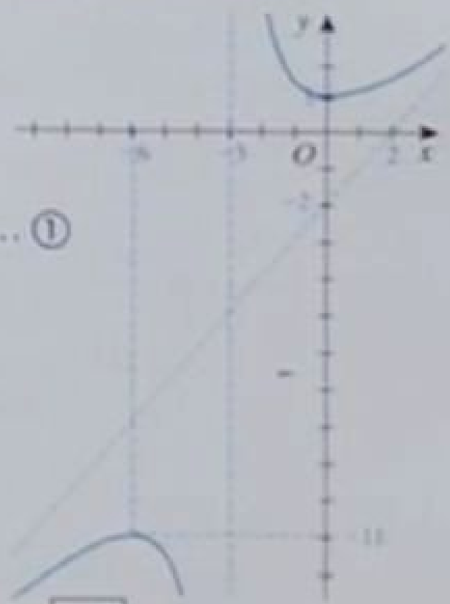
When studying fractional functions, we can easily obtain the equations of the asymptotes if we can separate the fractions:

$$y = \frac{x^2}{x+1} = \underbrace{x-1}_{y = x-1} + \underbrace{\frac{1}{x+1}}_{x = -1}$$

(Note how the asymptotes are derived from this format of the function.)

For each given fractional function, create a variation table and note where the function increases and decreases. Then, state the relative extreme value(s) and asymptote(s), and draw the graph.

(1) $y = \dfrac{x^2+x+3}{x+3}$



[Sol] $y = \dfrac{x^2+x+3}{x+3} = \boxed{x-2} + \dfrac{\boxed{9}}{x+3}$ …①

$y' = 1 - \dfrac{9}{(x+3)^2} = \dfrac{x(x+6)}{(x+3)^2}$

x	…	-6	…	-3	…	0	…
y'	+	0	−		−	0	+
y	↗	-11	↘		↘	1	↗

There is a relative maximum value of $\boxed{-11}$, at $x = \boxed{-6}$.

There is a relative minimum value of $\boxed{1}$, at $x = \boxed{0}$.

From ①,
The asymptotes are: $\boxed{x = -3}$, $\boxed{y = x - 2}$

O 21 b

(2) $y = \dfrac{x^2 - 3x}{x^2 + 3}$

[Sol] $y = 1 - \dfrac{3x + 3}{x^2 + 3} = 1 - \dfrac{3(x+1)}{x^2+3}$ $\quad \cdots$ ①

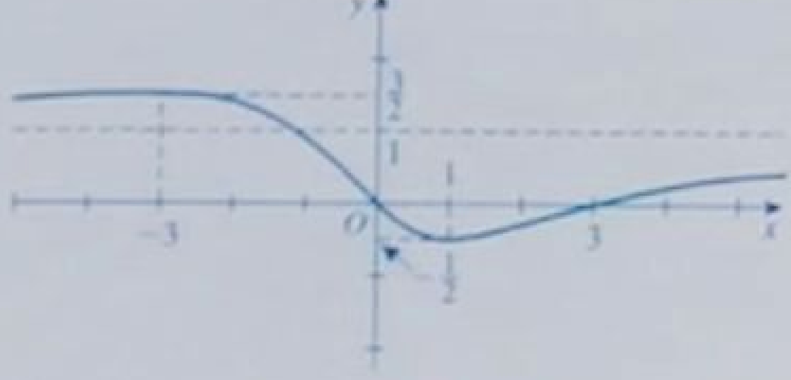
$$y' = -3 \cdot \frac{x^2 + 3 - (x+1) \cdot 2x}{(x^2+3)^2} = \frac{3(x-1)(x+3)}{(x^2+3)^2}$$

x	$\cdots$	-3	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	$\dfrac{3}{2}$	↘	$-\dfrac{1}{2}$	↗

There is a relative maximum value of $\dfrac{3}{2}$, at $x = -3$.

There is a relative minimum value of $-\dfrac{1}{2}$, at $x = 1$.

From ①,
The asymptote is: $y = 1$



Note: A curve may cross an asymptote. As x approaches positive or negative infinity, the curve approaches that asymptote.

Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

For each given fractional function, create a variation table and note where the function increases and decreases. Then, state the relative extreme value(s) and asymptote(s), and draw the graph.

(1) $y = \dfrac{x^2+1}{x}$

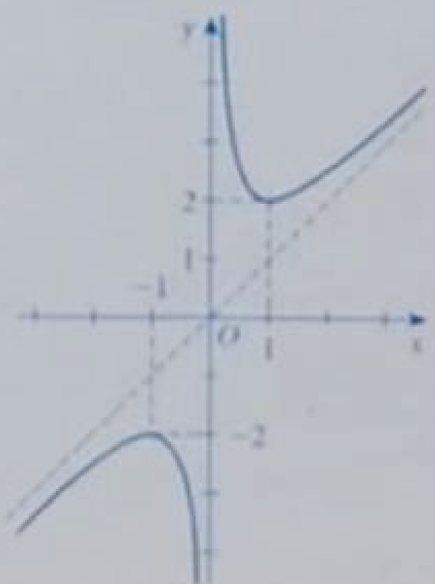
[Sol] $y = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}$ …①

$y' = 1 - \dfrac{1}{x^2} = \dfrac{(x+1)(x-1)}{x^2}$

x	…	-1	…	0	…	1	…
y'	$+$	0	$-$		$-$	0	$+$
y	↗	-2	↘		↘	2	↗

There is a relative maximum value of -2, at $x = -1$.
There is a relative minimum value of 2, at $x = 1$.

From ①,
The asymptotes are: $y = x$, $x = 0$



O 22 b

(2) $y = \dfrac{4x - 10}{x^2 - 4}$

[Sol] $y' = -\dfrac{4(x-1)(x-4)}{(x^2-4)^2}$

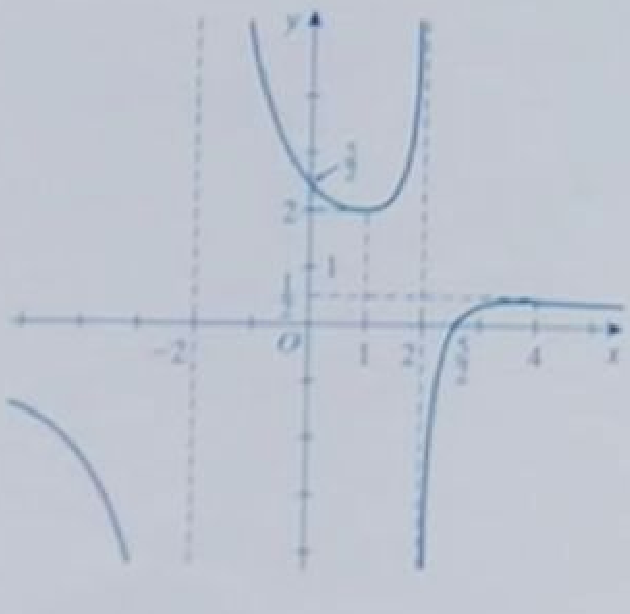
x	$\cdots$	-2	$\cdots$	1	$\cdots$	2	$\cdots$	4	$\cdots$
y'	$-$		$-$	0	$+$		$+$	0	$-$
y	$\searrow$		$\searrow$	2	$\nearrow$		$\nearrow$	$\frac{1}{2}$	$\searrow$

There is a relative maximum value of $\boxed{\frac{1}{2}}$, at $x = \boxed{4}$.

There is a relative minimum value of $\boxed{2}$, at $x = \boxed{1}$.

Since $\lim\limits_{x \to -2^{\pm}} y = \boxed{\pm\infty}$, $\lim\limits_{x \to 2^{\pm}} y = \boxed{\mp\infty}$, and $\lim\limits_{x \to \pm\infty} y = \boxed{0}$.

The asymptotes are: $x = \pm 2$, $y = 0$



O 23 a Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

For each given function, create a variation table and note where the function increases and decreases. Then, state the relative extreme value(s) and asymptote(s), (if any), and draw the graph.

(1) $y = \dfrac{x+1}{\sqrt{x^2+3}}$

[Sol] $y' = \dfrac{\sqrt{x^2+3} - (x+1)\cdot\frac{1}{2}(x^2+3)^{-\frac{1}{2}}\cdot 2x}{x^2+3} = \dfrac{3-x}{\sqrt{(x^2+3)^3}}$

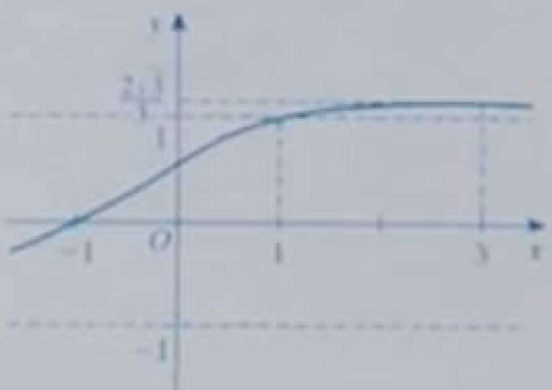
x	$\cdots$	3	$\cdots$
y'	+	0	−
y	↗	$\dfrac{2\sqrt{3}}{3}$	↘

There is a relative maximum value of $\dfrac{2\sqrt{3}}{3}$, at $x = 3$.

There is no relative minimum value.

Since $\lim\limits_{x\to-\infty} y = -1$ and $\lim\limits_{x\to+\infty} y = 1$,

The asymptotes are: $y = -1$, $y = 1$



O 23 b

(2) $y = x\sqrt{1-x^2}$ (Since $1 - x^2 \geq 0$, the domain is $-1 \leq x \leq 1$.)

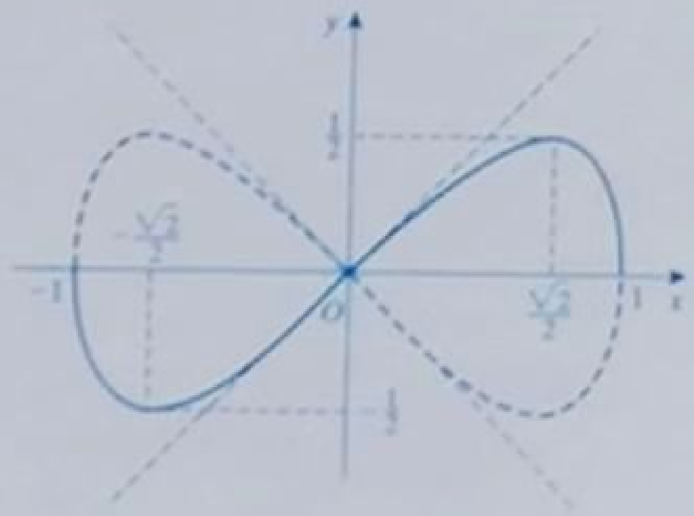
[Sol] $y' = \sqrt{1-x^2} + x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) = -2 \cdot \dfrac{\left(x + \frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right)}{\sqrt{1-x^2}}$

x	-1	$\cdots$	$-\frac{\sqrt{2}}{2}$	$\cdots$	$\frac{\sqrt{2}}{2}$	$\cdots$	1
y'	/	$-$	0	$+$	0	$-$	/
y	0	$\searrow$	$-\frac{1}{2}$	$\nearrow$	$\frac{1}{2}$	$\searrow$	0

There is a relative maximum value of $\frac{1}{2}$, at $x = \frac{\sqrt{2}}{2}$.

There is a relative minimum value of $-\frac{1}{2}$, at $x = -\frac{\sqrt{2}}{2}$.

There are no asymptotes.



Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

For each given function, create a variation table and note where the function increases and decreases. Then, state the relative extreme values and draw the graph.

(1) $y = \sin x(1+\cos x) \qquad (0 \leq x \leq 2\pi)$

[Sol] $y' = \cos x(1+\cos x) + \sin x \cdot (-\sin x)$

$= 2\cos^2 x + \cos x - 1$

$= (\cos x + 1)(2\cos x - 1)$

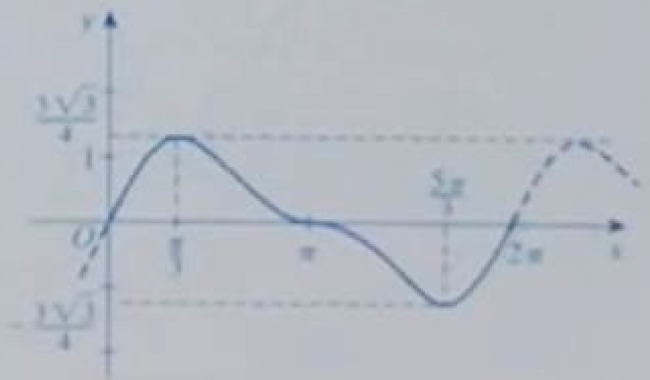
From $\cos x = -1$, $\quad x = \pi$

From $\cos x = \dfrac{1}{2}$, $\quad x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

x	0	$\cdots$	$\dfrac{\pi}{3}$	$\cdots$	π	$\cdots$	$\dfrac{5\pi}{3}$	$\cdots$	2π
y'	+	+	0	−	0	−	0	+	+
y	0	↗	$\dfrac{3\sqrt{3}}{4}$	↘	0	↘	$-\dfrac{3\sqrt{3}}{4}$	↗	0

There is a relative maximum value of $\dfrac{3\sqrt{3}}{4}$, at $x = \dfrac{\pi}{3}$.

There is a relative minimum value of $-\dfrac{3\sqrt{3}}{4}$, at $x = \dfrac{5\pi}{3}$.



O 24 b

(2) $y = 2\sin x - \sin 2x \qquad (0 \leq x \leq 2\pi)$

[Sol] $y' = 2\cos x - 2\cos 2x$

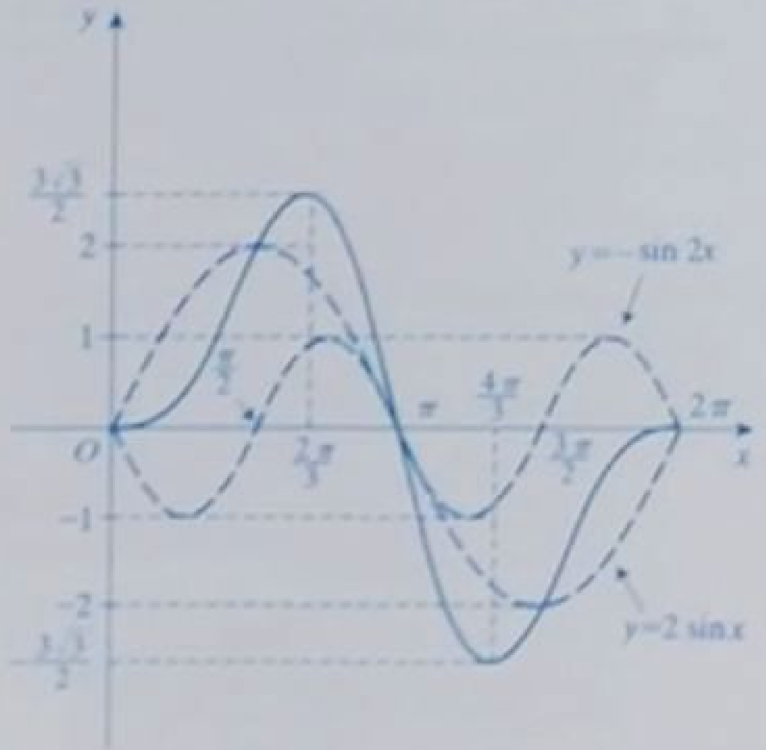
$= 4\sin\dfrac{x}{2}\sin\dfrac{3}{2}x$

From the Sum-to-Product Formulas (M124a)

x	0	$\cdots$	$\frac{2\pi}{3}$	$\cdots$	$\frac{4\pi}{3}$	$\cdots$	2π
y'	0	$+$	0	$-$	0	$+$	0
y	0	↗	$\frac{3\sqrt{3}}{2}$	↘	$-\frac{3\sqrt{3}}{2}$	↗	0

There is a relative maximum value of $\dfrac{3\sqrt{3}}{2}$, at $x = \dfrac{2\pi}{3}$.

There is a relative minimum value of $-\dfrac{3\sqrt{3}}{2}$, at $x = \dfrac{4\pi}{3}$.



Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	–

For each given function, create a variation table and note where the function increases and decreases. Then, state the relative extreme value(s) and asymptote(s), (if any), and draw the graph.

(1) $y = 2x - \tan x \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

[Sol] $y = 2x - \dfrac{\sin x}{\boxed{\cos x}}$

$$y' = 2 - \frac{1}{\cos^2 x} = \frac{2\left(\cos x + \frac{1}{\sqrt{2}}\right)\left(\cos x - \frac{1}{\sqrt{2}}\right)}{\cos^2 x}$$

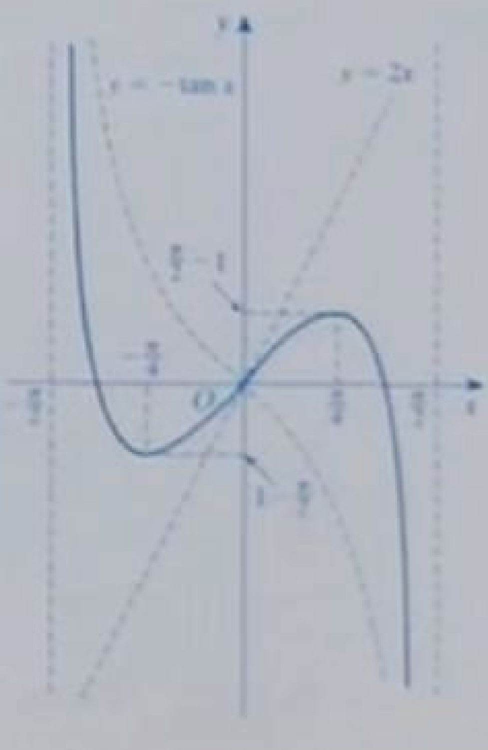
x	$-\frac{\pi}{2}$	$\cdots$	$-\frac{\pi}{4}$	$\cdots$	$\frac{\pi}{4}$	$\cdots$	$\frac{\pi}{2}$
y'		$-$	0	$+$	0	$-$	
y		$\searrow$	$1-\frac{\pi}{2}$	$\nearrow$	$\frac{\pi}{2}-1$	$\searrow$	

There is a relative maximum value of $\frac{\pi}{2} - 1$, at $x = \frac{\pi}{4}$.

There is a relative minimum value of $1 - \frac{\pi}{2}$, at $x = -\frac{\pi}{4}$.

(Since $\lim\limits_{x \to -\frac{\pi}{2}} y = +\infty$ and $\lim\limits_{x \to \frac{\pi}{2}} y = -\infty$,

The asymptotes are: $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$)



O 25 b

(2) $y = x\sin x + \cos x \qquad (0 \le x \le 2\pi)$

[Sol] $y' = \sin x + x\cos x - \sin x$

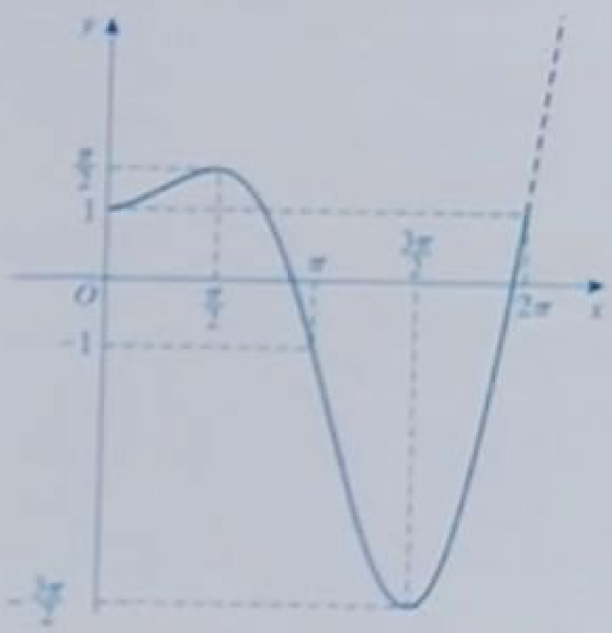
$\qquad = x\cos x$

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	$\frac{3\pi}{2}$	$\cdots$	2π
y'	0	$+$	0	$-$	0	$+$	$+$
y	1	$\nearrow$	$\frac{\pi}{2}$	$\searrow$	$-\frac{3\pi}{2}$	$\nearrow$	1

There is a relative maximum value of $\frac{\pi}{2}$, at $x = \frac{\pi}{2}$.

There is a relative minimum value of $-\frac{3\pi}{2}$, at $x = \frac{3\pi}{2}$.

There are no asymptotes.



Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

For each given function, create a variation table and note where the function increases and decreases. Then, state the relative extreme value(s) and asymptote(s), (if any), and draw the graph.

(1) $y = \dfrac{4x^2 - 4x + 1}{4(x-1)}$

[Sol] $y = \dfrac{4x^2 - 4x + 1}{4(x-1)} = x + \dfrac{1}{4(x-1)}$ $\cdots$ ①

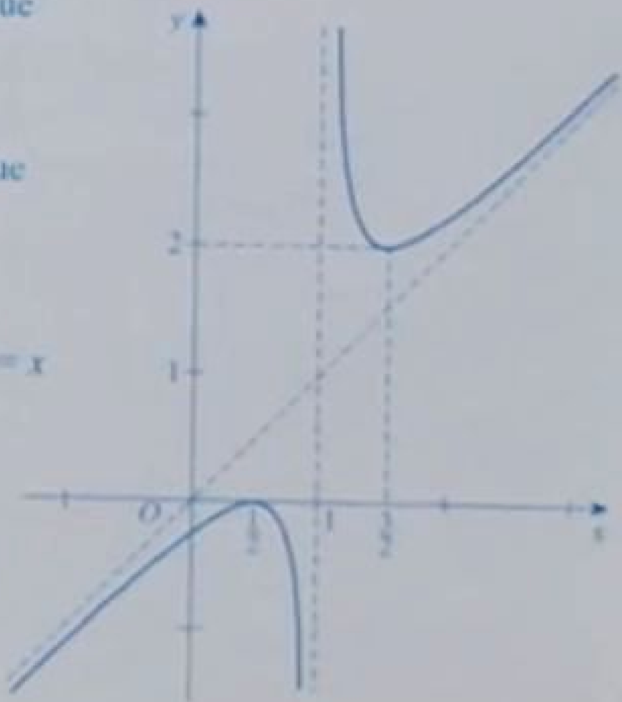
$$y' = 1 - \frac{1}{4(x-1)^2} = \frac{(x-1)^2 - \frac{1}{4}}{(x-1)^2}$$

x	$\cdots$	$\frac{1}{2}$	$\cdots$	1	$\cdots$	$\frac{3}{2}$	$\cdots$
y'	+	0	–		–	0	+
y	↗	0	↘		↘	2	↗

There is a relative maximum value of 0, at $x = \dfrac{1}{2}$.

There is a relative minimum value of 2, at $x = \dfrac{3}{2}$.

From ①,
The asymptotes are: $x = 1$, $y = x$



O 26 b

(2) $y = \dfrac{4x^2 + 2x + 4}{x^2 + 1}$

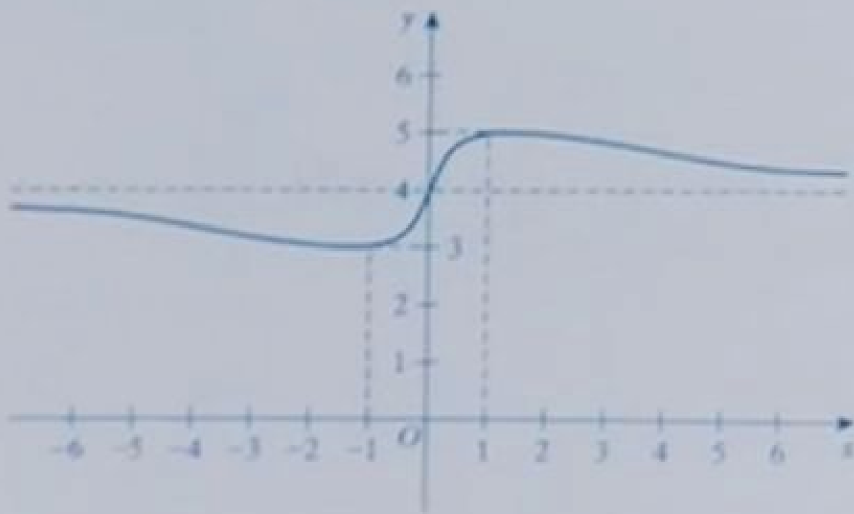
[Sol] $y = 4 + \dfrac{2x}{x^2 + 1}$ $\quad \cdots ①$

$$y' = -2 \cdot \frac{(x+1)(x-1)}{(x^2+1)^2}$$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
y'	$-$	0	$+$	0	$-$
y	$\searrow$	3	$\nearrow$	5	$\searrow$

There is a relative maximum value of 5, at $x = 1$.
There is a relative minimum value of 3, at $x = -1$.

From ①.
The asymptote is: $y = 4$



Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

For each given function, create a variation table and note where the function increases and decreases. Then, state the relative extreme value(s) and asymptote(s), (if any), and draw the graph.

(1) $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

[Sol] $y = \dfrac{x^2 - x + 1}{x^2 + x + 1} = 1 - \dfrac{2x}{x^2 + x + 1}$ ···①

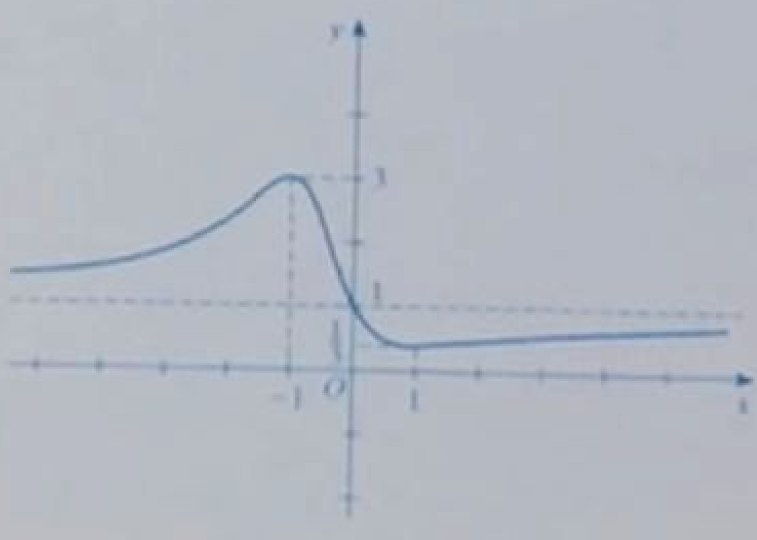
$y' = 2 \cdot \dfrac{(x+1)(x-1)}{(x^2 + x + 1)^2}$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	3	↘	$\frac{1}{3}$	↗

There is a relative maximum value of 3, at $x = -1$.

There is a relative minimum value of $\frac{1}{3}$, at $x = 1$.

From ①,
The asymptote is: $y = 1$



O 27 b

(2) $y = x + \sin 2x \qquad (-\pi \le x \le \pi)$

[Sol] $y' = 1 + 2\cos 2x \qquad (-2\pi \le 2x \le 2\pi)$

From $\qquad \cos 2x = -\dfrac{1}{2},$

$$2x = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$$

Therefore, $\qquad x = -\dfrac{2\pi}{3}, -\dfrac{\pi}{3}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

x	$\cdots$	$-\frac{2\pi}{3}$	$\cdots$	$-\frac{\pi}{3}$	$\cdots$	$\frac{\pi}{3}$	$\cdots$	$\frac{2\pi}{3}$	$\cdots$
y'	+	0	−	0	+	0	−	0	+
y	↗	$\frac{\sqrt{3}}{2}-\frac{2\pi}{3}$	↘	$-\frac{\sqrt{3}}{2}-\frac{\pi}{3}$	↗	$\frac{\sqrt{3}}{2}+\frac{\pi}{3}$	↘	$-\frac{\sqrt{3}}{2}+\frac{2\pi}{3}$	↗

The relative maximum values are:

$\dfrac{\sqrt{3}}{2} - \dfrac{2\pi}{3}$, at $x = -\dfrac{2\pi}{3}$, and

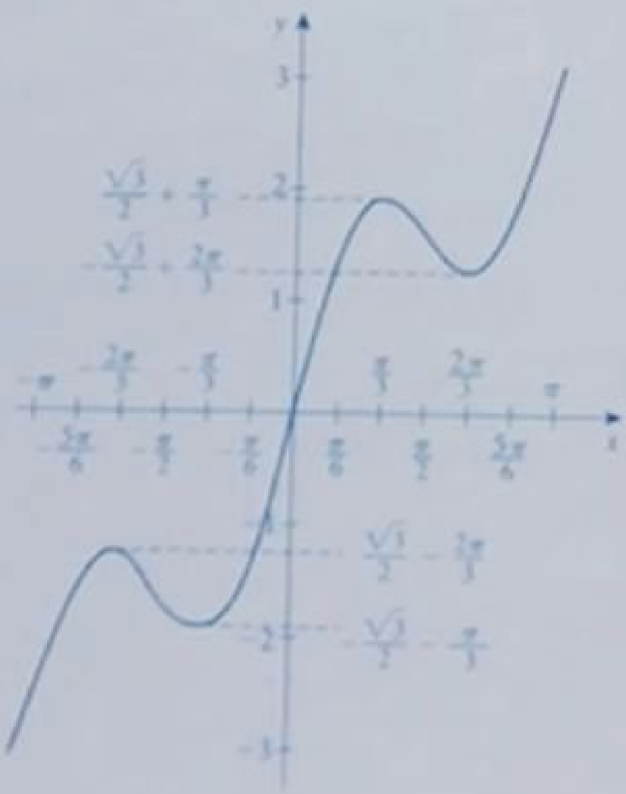
$\dfrac{\sqrt{3}}{2} + \dfrac{\pi}{3}$, at $x = \dfrac{\pi}{3}$.

The relative minimum values are:

$-\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{3}$, at $x = -\dfrac{\pi}{3}$, and

$-\dfrac{\sqrt{3}}{2} + \dfrac{2\pi}{3}$, at $x = \dfrac{2\pi}{3}$.

There are no asymptotes.



KUMON

O 28 a Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	–

For each given function, create a variation table and note where the function increases and decreases. Then, state the relative extreme value(s) and asymptote(s), (if any), and draw the graph.

(1) $y = x^2e^{-x}$ (where $e \approx 2.718$) (Note: $\lim\limits_{x \to +\infty} \frac{x^2}{e^x} = 0$)

[Sol] $y' = 2xe^{-x} + x^2(-1)e^{-x} = e^{-x}x(2-x)$

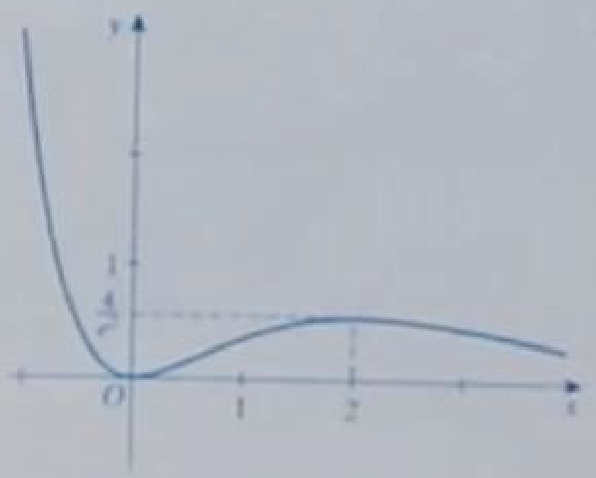
x	$\cdots$	0	$\cdots$	2	$\cdots$
y'	$-$	0	$+$	0	$-$
y	$\searrow$	0	$\nearrow$	$\frac{4}{e^2}$	$\searrow$

There is a relative maximum value of $\frac{4}{e^2} \approx 0.54$, at $x = 2$

There is a relative minimum value of 0, at $x = 0$.

Since $\lim\limits_{x \to +\infty} y = \lim\limits_{x \to +\infty} \frac{x^2}{e^x} = 0$.

The asymptote is: $y = 0$



O 28 b

(2) $y=\dfrac{\ln x}{x}$ (where $x>0$) (Note: $\lim_{x\to+\infty}\dfrac{\ln x}{x}=0$)

[Sol] $y'=\dfrac{\frac{1}{x}\cdot x-\ln x}{x^2}=\dfrac{1-\ln x}{x^2}=\dfrac{\ln e-\ln x}{x^2}$

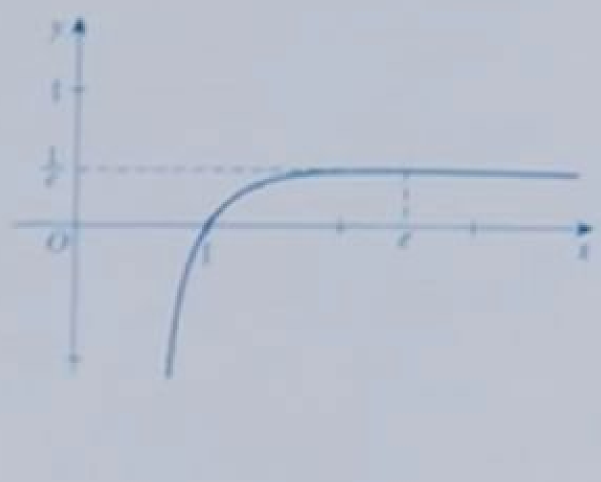
x	$\cdots$	e	$\cdots$
y'	$+$	0	$-$
y	$\nearrow$	$\frac{1}{e}$	$\searrow$

There is a relative maximum value of $\dfrac{1}{e}\approx 0.37$, at $x=e$.

There is no relative minimum value.

Since $\lim_{x\to 0^+} y=-\infty$ and $\lim_{x\to+\infty} y=\lim_{x\to+\infty}\dfrac{\ln x}{x}=0$,

The asymptotes are: $x=0$, $y=0$



Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

For each given function, create a variation table and note where the function increases and decreases. Then, state the relative extreme value(s) and asymptote(s), (if any), and draw the graph.

(1) $y = x^2 + \ln(2 - x^2)$

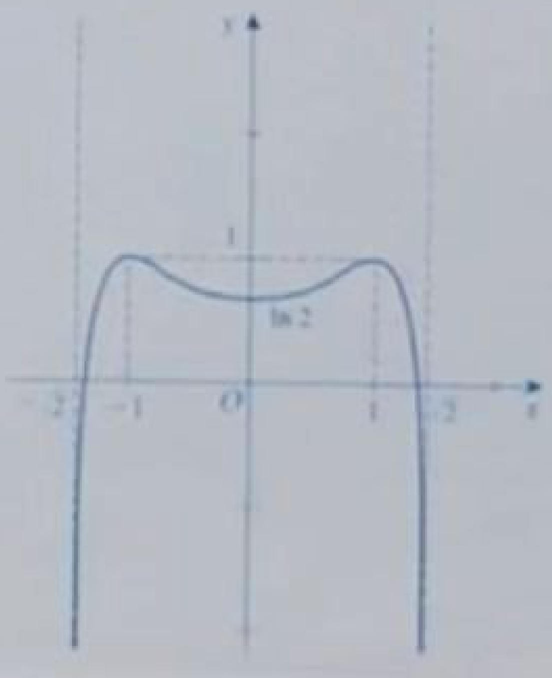
[Sol] $y' = 2x - \dfrac{2x}{2-x^2} = \dfrac{2x(1-x^2)}{2-x^2}$

x	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	0	$+$	0	$-$
y	↗	1	↘	$\ln 2$	↗	1	↘

There is a relative maximum value of 1, at $x = \pm 1$.
There is a relative minimum value of $\ln 2$, at $x = 0$.

Since $\lim\limits_{x \to -\sqrt{2}} y = -\infty$ and $\lim\limits_{x \to \sqrt{2}} y = -\infty$,

The asymptotes are: $x = \pm\sqrt{2}$



O 29 b

(2) $y = x^2 - 5x + 5 + 2\ln x$

[Sol] $y' = 2x - 5 + \dfrac{2}{x} = \dfrac{(2x-1)(x-2)}{x}$

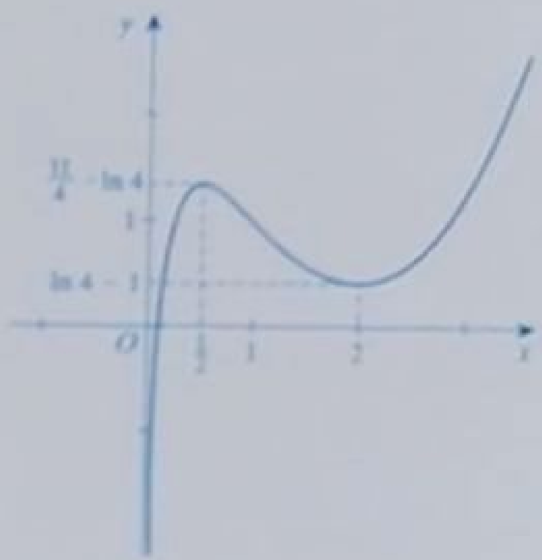
x	0	$\cdots$	$\frac{1}{2}$	$\cdots$	2	$\cdots$
y'		$+$	0	$-$	0	$+$
y		$\nearrow$	$\frac{11}{4} - \ln 4$	$\searrow$	$\ln 4 - 1$	$\nearrow$

There is a relative maximum value of $\dfrac{11}{4} - \ln 4$, at $x = \dfrac{1}{2}$.

There is a relative minimum value of $\ln 4 - 1$, at $x = 2$.

Since $\lim\limits_{x \to +0} y = -\infty$,

The asymptote is: $x = 0$



Increasing and Decreasing Functions

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

For each given function, create a variation table and note where the function increases and decreases. Then, state the relative extreme value(s) and asymptote(s), (if any), and draw the graph.

(1) $y = \dfrac{4x+3}{x^2+1}$

[Sol] $y' = -2 \cdot \dfrac{(2x^2+3x-2)}{(x^2+1)^2} = -2 \cdot \dfrac{(2x-1)(x+2)}{(x^2+1)^2}$

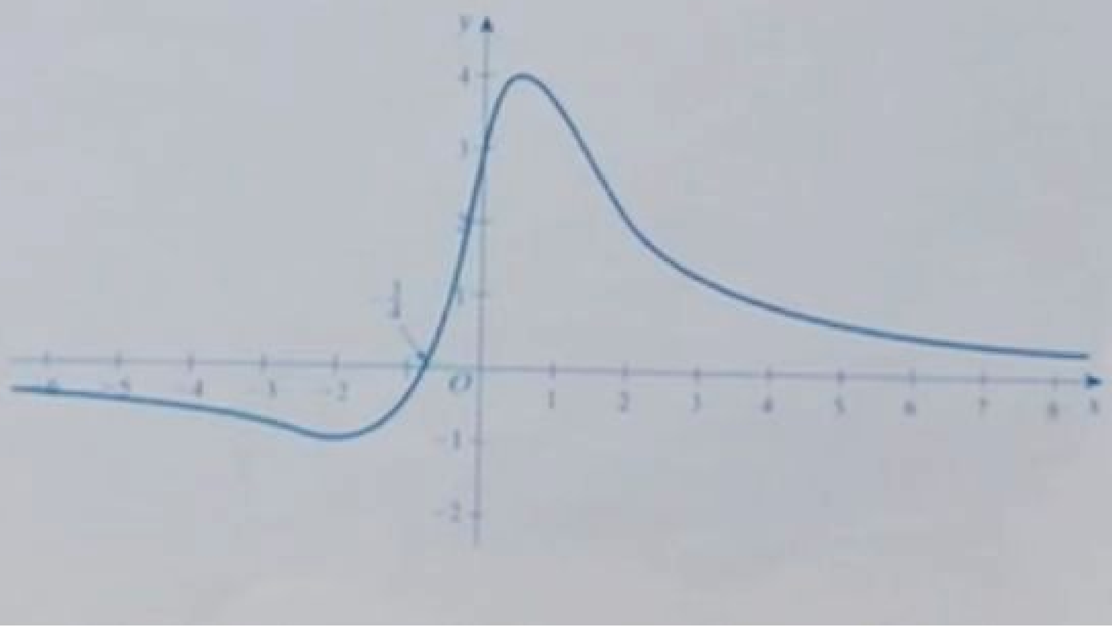
x	$\cdots$	-2	$\cdots$	$\frac{1}{2}$	$\cdots$
y'	$-$	0	$+$	0	$-$
y	↘	-1	↗	4	↘

There is a relative maximum value of 4, at $x = \dfrac{1}{2}$

There is a relative minimum value of -1, at $x = -2$.

Since $\lim\limits_{x \to \pm\infty} y = 0$,

The asymptote is: $y = 0$



O 30 b

(2) $y=\sin x-\dfrac{1}{2}\sin 2x+\dfrac{1}{3}\sin 3x \qquad (-\pi \leq x \leq \pi)$

[Sol] $y'=\cos x-\cos 2x+\cos 3x$

$$=(\cos x+\cos 3x)-\cos 2x$$
$$=2\cos 2x\cos x-\cos 2x$$
$$=2\cos 2x\left(\cos x-\frac{1}{2}\right)$$
$$=2(2\cos^2 x-1)\left(\cos x-\frac{1}{2}\right)$$
$$=4\left(\cos x+\frac{1}{\sqrt{2}}\right)\left(\cos x-\frac{1}{\sqrt{2}}\right)\left(\cos x-\frac{1}{2}\right)$$

Since the graph of y is symmetric with respect to the origin, creating the variation table over the domain $0\leq x\leq\pi$.

x	0	$\cdots$	$\frac{\pi}{4}$	$\cdots$	$\frac{\pi}{3}$	$\cdots$	$\frac{3\pi}{4}$	$\cdots$	π
y'	+	+	0	−	0	+	0	−	−
y	0	↗	$\frac{4\sqrt{2}-3}{6}$	↘	$\frac{\sqrt{3}}{4}$	↗	$\frac{4\sqrt{2}+3}{6}$	↘	0

Therefore, in the original domain $-\pi\leq x\leq\pi$,

The relative maximum values are:

$-\dfrac{\sqrt{3}}{4}$, at $x=-\dfrac{\pi}{3}$

$\dfrac{4\sqrt{2}-3}{6}$, at $x=\dfrac{\pi}{4}$

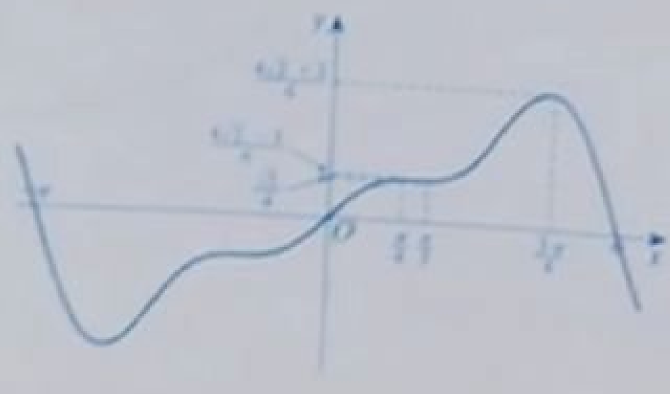
$\dfrac{4\sqrt{2}+3}{6}$, at $x=\dfrac{3\pi}{4}$

The relative minimum values are:

$\dfrac{3-4\sqrt{2}}{6}$, at $x=-\dfrac{\pi}{4}$

$-\dfrac{4\sqrt{2}+3}{6}$, at $x=-\dfrac{3\pi}{4}$

$\dfrac{\sqrt{3}}{4}$, at $x=\dfrac{\pi}{3}$



KUMON

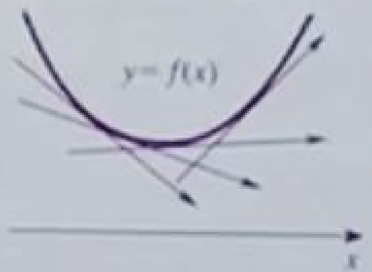
O 31 a   Concavity and Tangent Lines

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

When $f''(x)>0$, the curve of $f(x)$ opens upward, and is called **concave up**.
When $f''(x)<0$, the curve of $f(x)$ opens downward, and is called **concave down**.

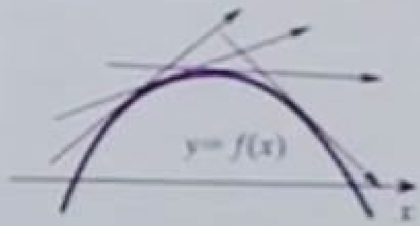
Concave up



Note that the slopes of the tangent lines $f'(x)$ uniformly increase.

$f''(x)>0$

Concave down



Note that the slopes of the tangent lines $f'(x)$ uniformly decrease.

$f''(x)<0$

1. In each exercise, determine whether the curve of the given function is concave up or concave down.

(1) $y=x^4+x^3+5x^2+x+6 \qquad (x>0)$

[Sol] $y'=\boxed{4x^3+3x^2+10x+1}$, $\quad y''=\boxed{12x^2+6x+10}$

Since $y''\ \boxed{>}\ 0$,

The curve is concave $\boxed{\text{up}}$ over $0<x<\infty$.

(2) $y=\ln x \qquad (x>0)$

[Sol] $y'=\dfrac{1}{x}$, $\quad y''=-\dfrac{1}{x^2}$

Since $y''<0$,

The curve is concave down over $0<x<\infty$.

(3) $y=e^x$

[Sol] $y'=e^x$, $\quad y''=e^x$

Since $y''>0$,

The curve is concave up.

(4) $y = \sin x \quad (-\pi < x < \pi)$

[Sol] $y' = \cos x, \quad y'' = -\sin x$

$$\begin{cases} \text{Over } -\pi < x < 0, \; y'' \boxed{>} 0, & \text{and the curve is concave } \boxed{\text{up}}. \\ \text{Over } \; 0 < x < \pi, \; y'' \boxed{<} 0, & \text{and the curve is concave } \boxed{\text{down}}. \end{cases}$$

In the vicinity of the origin, the curve of $y = \sin x$ is:

$$\begin{cases} \text{Concave up} & \text{when } x \boxed{<} 0, \text{ but changes to} \\ \text{Concave down} & \text{when } x \boxed{>} 0. \end{cases}$$

> The point at which the curvature changes is called an **inflection point**.
>
> Therefore, a change in the sign of $f''(x)$ implies an inflection point.

2. Obtain the inflection point(s) of the following functions.

(1) $y = x^4 - 4x^3 + 16x$

[Sol] $y' = 4x^3 - 12x^2 + 16, \quad y'' = 12x^2 - 24x$
$= 12x(x-2)$

Therefore, when $x = 0, \quad y = 0$
when $x = 2, \quad y = 16$

Inflection points: $(0, 0), (2, 16)$

(2) $y = x^3 + 3ax^2 + a^3 \quad (a > 0)$

[Sol] $y' = 3x^2 + 6ax, \quad y'' = 6(x + a)$

Therefore, when $x = -a, \quad y = 3a^3$

Inflection point: $(-a, 3a^3)$

Note: An inflection point is a point at which the concavity changes,
i.e. a point at which $f''(x) = 0$.

KUMON

O 32 a

Concavity and Tangent Lines

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

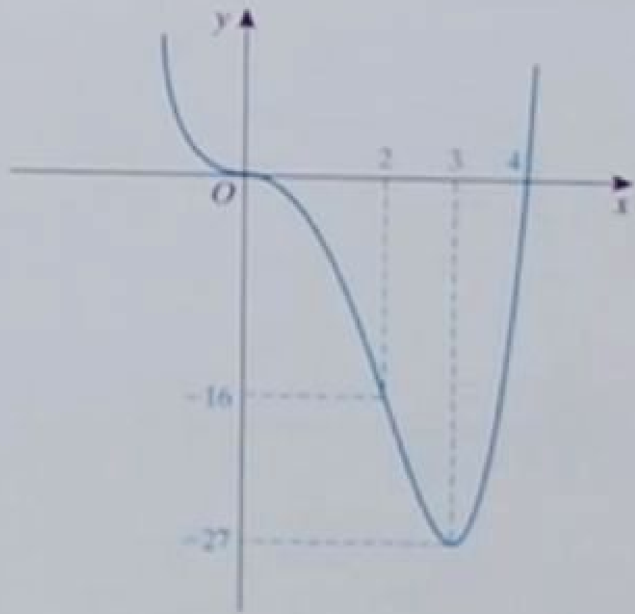
For each given function, create a variation table indicating where the curve is concave up and concave down, and note the point(s) of inflection. Then, state the asymptote(s), (if any), and draw the graph.

(1) $y = x^3(x-4)$

[Sol] $y' = 4x^3 - 12x^2 = 4x^2(x-3)$

$y'' = 12x(x-2)$

x	$\cdots$	0	$\cdots$	2	$\cdots$	3	$\cdots$
y'	$-$	0	$-$	$-$	$-$	0	$+$
y''	$+$	0	$-$	0	$+$	$+$	$+$
y		i.p.		i.p.		-27	



Note: means concave up and decreasing. means concave up and increasing.
means concave down and decreasing. means concave down and increasing.
i.p. is an abbreviation for inflection point.

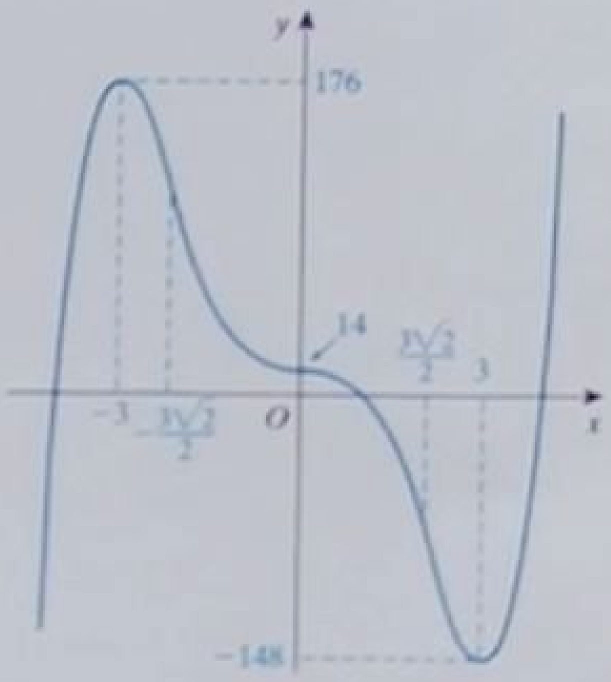
O 32 b

(2) $y = x^5 - 15x^3 + 14$

[Sol] $y' = 5x^4 - 45x^2 = 5x^2(x+3)(x-3)$

$$y'' = 20x^3 - 90x = 20x\left(x + \frac{3\sqrt{2}}{2}\right)\left(x - \frac{3\sqrt{2}}{2}\right)$$

x	$\cdots$	-3	$\cdots$	$-\frac{3\sqrt{2}}{2}$	$\cdots$	0	$\cdots$	$\frac{3\sqrt{2}}{2}$	$\cdots$	3	$\cdots$
y'	$+$	0	$-$	$-$	$-$	0	$-$	$-$	$-$	0	$+$
y''	$-$	$-$	$-$	0	$+$	0	$-$	0	$+$	$+$	$+$
y	↗	176	↘	i.p.	↘	i.p.	↘	i.p.	↘	-148	↗



Concavity and Tangent Lines

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

For each given function, create a variation table indicating where the curve is concave up and concave down, and note the point(s) of inflection. Then, state the asymptote(s), (if any), and draw the graph.

(1) $y = \dfrac{1}{1+x^2}$

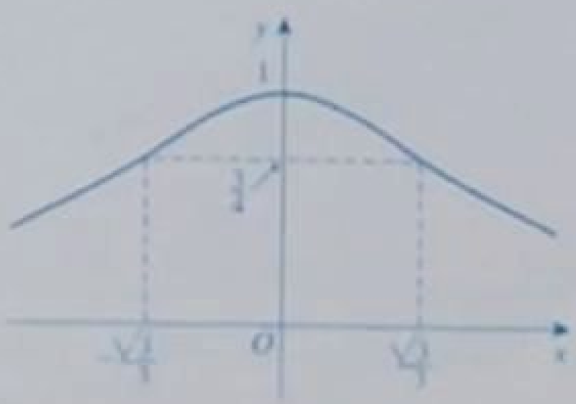
[Sol] $y' = -\dfrac{2x}{(1+x^2)^2}$

$y'' = \dfrac{2(3x^2-1)}{(1+x^2)^3}$

x	$\cdots$	$-\dfrac{\sqrt{3}}{3}$	$\cdots$	0	$\cdots$	$\dfrac{\sqrt{3}}{3}$	$\cdots$
y'	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	$+$	0	$-$	$-$	$-$	0	$+$
y	⤴	i.p.	⌒	1	⤵	i.p.	⤵

Since $\lim\limits_{x\to\pm\infty} y = 0$,

The asymptote is: $y = 0$



O 33 b

(2) $y = x^2 + \dfrac{1}{x}$

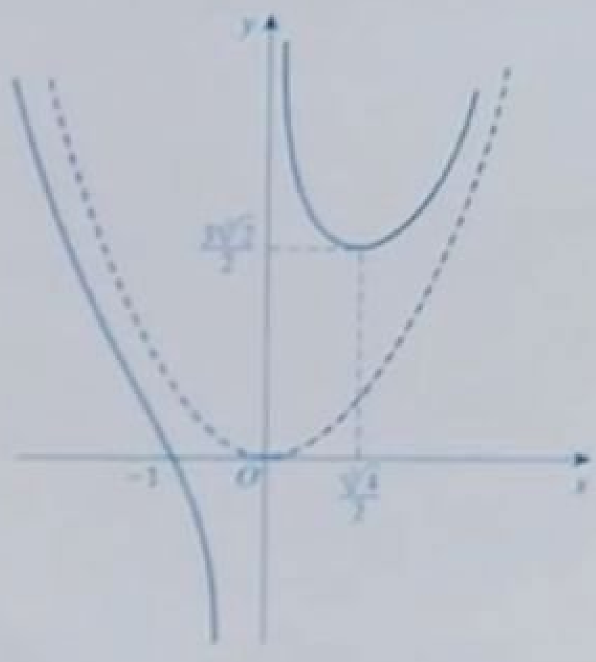
[Sol] $y' = 2x - \dfrac{1}{x^2} = \dfrac{1}{x^2}(2x^3 - 1)$

$$y'' = \frac{2(x^3+1)}{x^3}$$

x	$\cdots$	-1	$\cdots$	0	$\cdots$	$\dfrac{\sqrt[3]{4}}{2}$	$\cdots$
y'	$-$	$-$	$-$		$-$	0	$+$
y''	$+$	0	$-$		$+$	$+$	$+$
y	$\searrow$	i.p.	$\searrow$		$\searrow$	$\dfrac{3\sqrt[3]{2}}{2}$	$\nearrow$

Since $\lim\limits_{x \to 0^+} y = +\infty$ and $\lim\limits_{x \to 0^-} y = -\infty$,

The asymptote is: $x = 0$



Note: As $x \to \pm\infty$, the curve approaches but never touches or crosses the parabola $y = x^2$. However, since $y = x^2$ is not a line, it is not considered to be an asymptote.

O 34 a Concavity and Tangent Lines

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

For each given function, create a variation table indicating where the curve is concave up and concave down, and note the point(s) of inflection. Then, state the asymptote(s), (if any), and draw the graph.

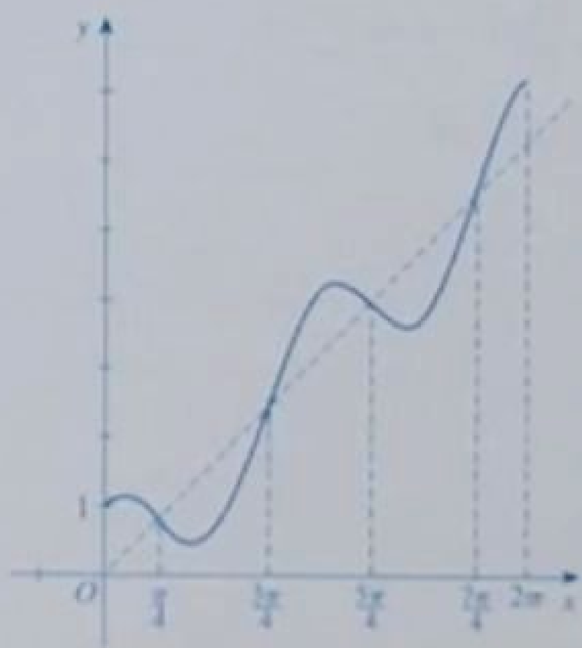
(1) $y = x + \cos 2x \quad (0 \leq x \leq 2\pi)$

[Sol] $y' = 1 - 2\sin 2x$

$y'' = -4\cos 2x$

x	0	$\cdots$	$\frac{\pi}{12}$	$\cdots$	$\frac{\pi}{4}$	$\cdots$	$\frac{5\pi}{12}$	$\cdots$	$\frac{3\pi}{4}$	$\cdots$	$\frac{13\pi}{12}$	$\cdots$	$\frac{5\pi}{4}$	$\cdots$	$\frac{17\pi}{12}$	$\cdots$	$\frac{7\pi}{4}$	$\cdots$	2π
y'	+	+	0	−	−	−	0	+	+	+	0	−	−	−	0	+	+	+	+
y''	−	−	−	−	0	+	+	+	0	−	−	−	0	+	+	+	0	−	−
y	1	↗	Rel. Max.	↘	I.P.	↘	Rel. Min.	↗	I.P.	↗	Rel. Max.	↘	I.P.	↘	Rel. Min.	↗	I.P.	↗	$2\pi+1$

$\frac{\pi}{12}+\frac{\sqrt{3}}{2}$ $\quad$ $\frac{5\pi}{12}-\frac{\sqrt{3}}{2}$ $\quad$ $\frac{13\pi}{12}+\frac{\sqrt{3}}{2}$ $\quad$ $\frac{17\pi}{12}-\frac{\sqrt{3}}{2}$



O 34 b

(2) $y = x^4 e^x$ $\left(\text{Note: } \lim\limits_{x\to+\infty}\dfrac{x^4}{e^x}=0\right)$

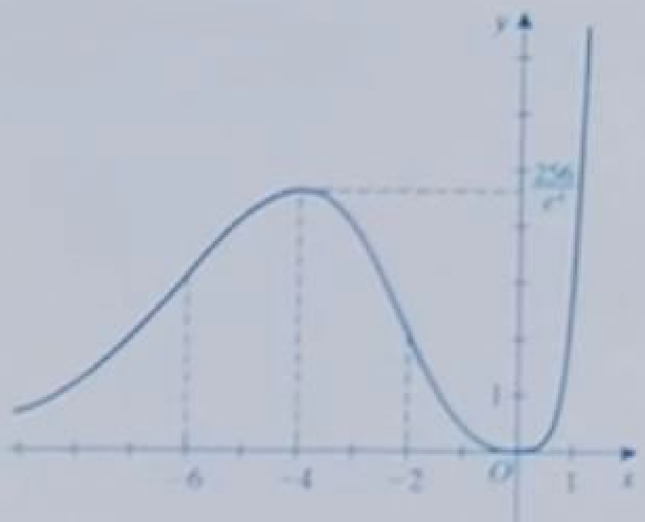
[Sol] $y' = e^x x^3(x+4)$

$y'' = e^x x^2(x+2)(x+6)$

x	$\cdots$	-6	$\cdots$	-4	$\cdots$	-2	$\cdots$	0	$\cdots$
y'	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
y''	$+$	0	$-$	$-$	$-$	0	$+$	0	$+$
y		i.p.		$\dfrac{256}{e^4}$		i.p.		0	

Since $\lim\limits_{x\to-\infty} y = \lim\limits_{t\to+\infty}\dfrac{t^4}{e^t}=0$,

The asymptote is: $y=0$



Concavity and Tangent Lines

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

For each given function, create a variation table indicating where the curve is concave up and concave down, and note the point(s) of inflection. Then, state the asymptote(s), (if any), and draw the graph.

(1) $y=\dfrac{x^2}{x^2+3}$

[Sol] $y=1-\dfrac{3}{x^2+3}$

$$y'=\frac{6x}{(x^2+3)^2}$$

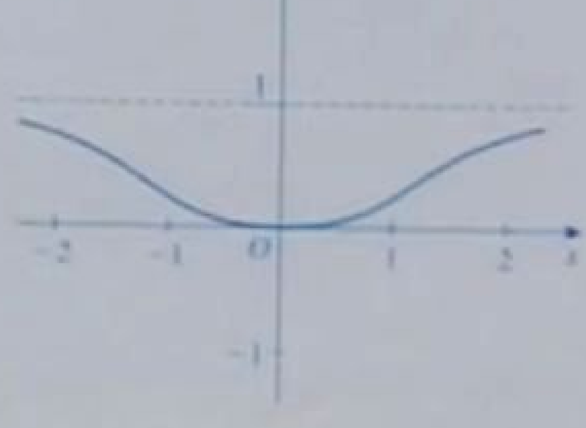
$$y''=6\left[\frac{(x^2+3)^2-4x^2(x^2+3)}{(x^2+3)^4}\right]$$

$$=6\left[\frac{-3x^2+3}{(x^2+3)^3}\right]=-\frac{18(x+1)(x-1)}{(x^2+3)^3}$$

x	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$
y'	$-$	$-$	$-$	0	$+$	$+$	$+$
y''	$-$	0	$+$	$+$	$+$	0	$-$
y		i.p.		0		i.p.	

Rel. min. at $x=0$

The asymptote is: $y=1$



O 35 b

(2) $y = \dfrac{x^3}{3(x+1)^2}$

[Sol] $y = \dfrac{x^3}{3(x+1)^2}$

$$3y = \frac{x^3}{(x+1)^2} = x - 2 + \frac{3x+2}{(x+1)^2} \qquad \cdots ①$$

$$3y' = 1 + \frac{3(x+1)^2 - (3x+2)\cdot 2(x+1)}{(x+1)^4}$$

$$= \frac{x^2(x+3)}{(x+1)^3}$$

$$3y'' = \frac{(3x^2+6x)(x+1)^3 - 3x^2(x+3)(x+1)^2}{(x+1)^6}$$

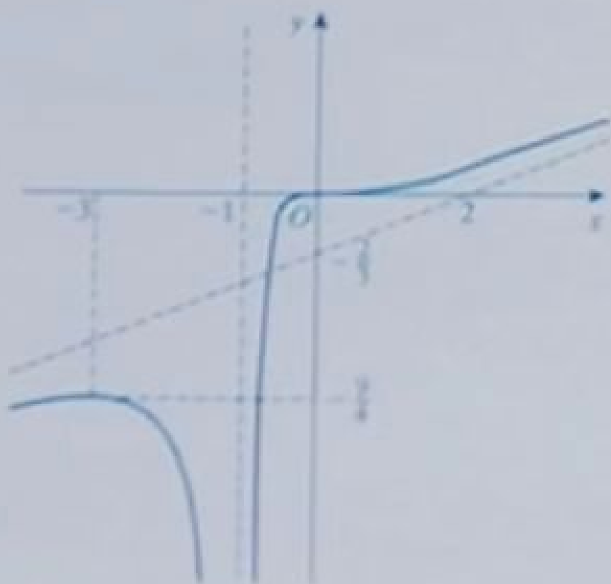
$$= \frac{6x}{(x+1)^4}$$

x	$\cdots$	-3	$\cdots$	-1	$\cdots$	0	$\cdots$
y'	$+$	0	$-$		$+$	0	$+$
y''	$-$	$-$	$-$		$-$	0	$+$
y		$-\dfrac{9}{4}$				i.p., 0	

From ①,

The asymptotes are:

$x = -1, \quad y = \dfrac{1}{3}(x-2)$



KUMON

O 36 a

Concavity and Tangent Lines

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

For each given function, create a variation table indicating where the curve is concave up and concave down, and note the point(s) of inflection. Then, state the asymptote(s), (if any), and draw the graph.

(1) $y=\dfrac{1}{\sqrt{1+x^2}}$

[Sol] $y=\dfrac{1}{\sqrt{1+x^2}}=(1+x^2)^{-\frac{1}{2}}$

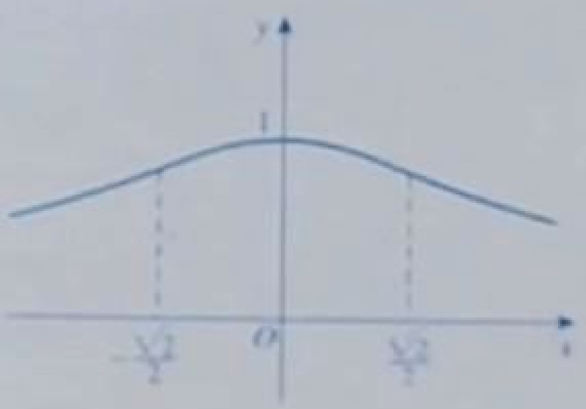
$$y'=-\frac{1}{2}(1+x^2)^{-\frac{3}{2}}\cdot 2x=-\frac{x}{\sqrt{(1+x^2)^3}}$$

$$y''=\frac{2x^2-1}{\sqrt{(1+x^2)^5}}=\frac{2\left(x+\frac{1}{\sqrt{2}}\right)\left(x-\frac{1}{\sqrt{2}}\right)}{\sqrt{(1+x^2)^5}}$$

x	…	$-\frac{1}{\sqrt{2}}$	…	0	…	$\frac{1}{\sqrt{2}}$	…
y'	+	+	+	0	−	−	−
y''	+	0	−	−	−	0	+
y	⤴	i.p., $\frac{\sqrt{6}}{3}$	⌒	1	⤵	i.p., $\frac{\sqrt{6}}{3}$	⤷

Since $\lim\limits_{x\to\pm\infty} y=0$,

The asymptote is: $y=0$



(Note: The curve is symmetric with respect to the y-axis.)

O 36 b

(2) $y = \dfrac{x}{\ln x}$

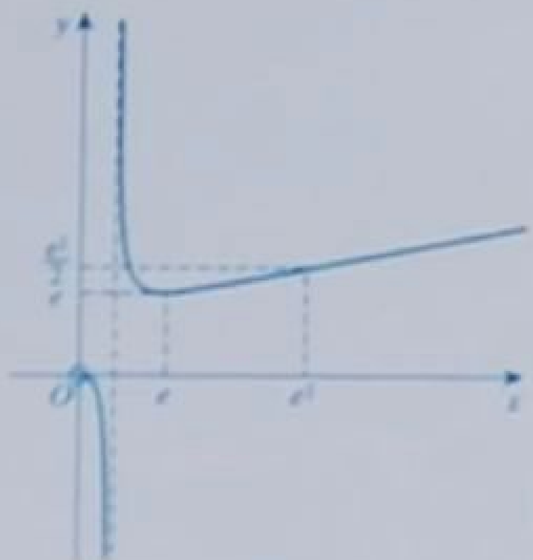
[Sol] $y' = \dfrac{\ln x - 1}{(\ln x)^2}$

$$y'' = \frac{\frac{1}{x}\ln x - 2\cdot\frac{1}{x}(\ln x - 1)}{(\ln x)^3} = \frac{2 - \ln x}{x(\ln x)^3}$$

x	0	$\cdots$	1	$\cdots$	e	$\cdots$	e^2	$\cdots$
y'		$-$		$-$	0	$+$	$+$	$+$
y''		$-$		$+$	$+$	$+$	0	$-$
y		$\searrow$		$\searrow$	e	$\nearrow$	i.p., $\dfrac{e^2}{2}$	$\nearrow$

Since $\lim\limits_{x\to 0^+} y = 0$ and $\lim\limits_{x\to 1^\pm} y = \pm\infty$,

The asymptote is: $x = 1$



KUMON

O 37 a　Concavity and Tangent Lines

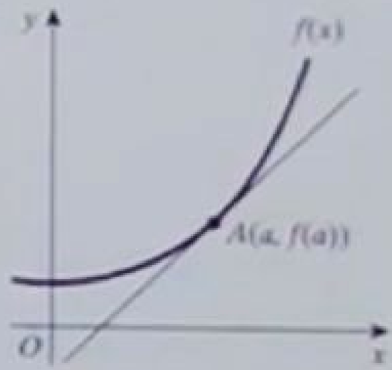
Time　:　to　:　Date　Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Given a curve with point $A(a, f(a))$, its slope at A is $f'(a)$.

The equation of the tangent line at point A can be expressed as:

$$y - \boxed{f(a)} = \boxed{f'(a)}\,(x - \boxed{a})$$



Answers: $f(a)$, $f'(a)$, a

Find the equation of the tangent line to each of the following curves at the given point. Then, graph the curve and the tangent line.

(1) $y = x^2 + 2x - 1$　　$(0, -1)$

[Sol] Letting $f(x) = x^2 + 2x - 1$,

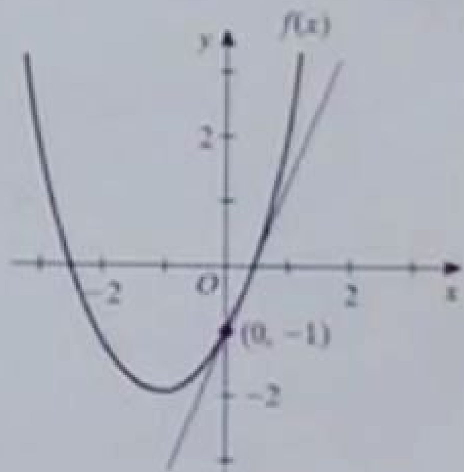
$f'(x) = 2x + 2$

$f'(0) = 2$

The equation of the tangent line is:

$y - (-1) = 2(x - 0)$

$y = 2x - 1$



(2) $y = -x^3 - x^2 + 2x$　　$(-1, -2)$

[Sol] Letting $f(x) = -x^3 - x^2 + 2x$,

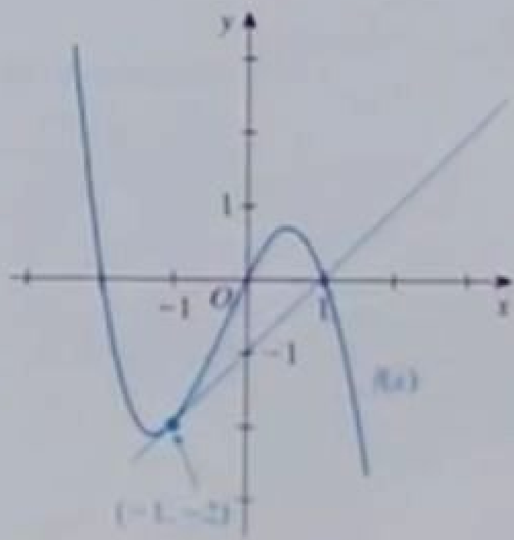
$f'(x) = -3x^2 - 2x + 2$

$f'(-1) = 1$

The equation of the tangent line is:

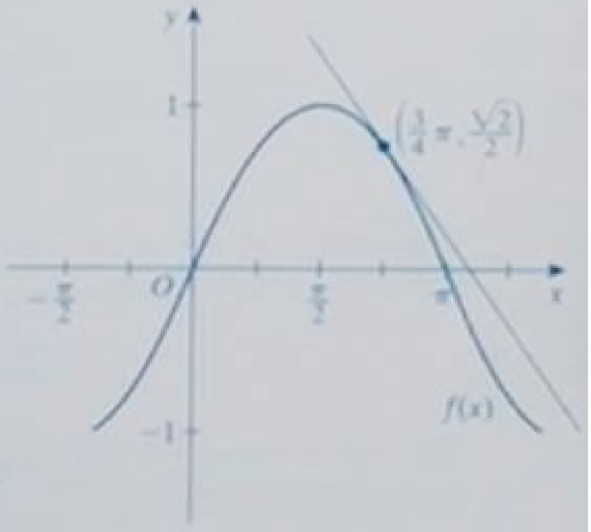
$y - (-2) = 1(x + 1)$

$y = x - 1$



O 37 b

(3) $y = \sin x$ $\qquad\qquad \left(\frac{3}{4}\pi, \frac{\sqrt{2}}{2}\right)$



[Sol] Letting $\quad f(x) = \sin x,$

$$f'(x) = \cos x$$

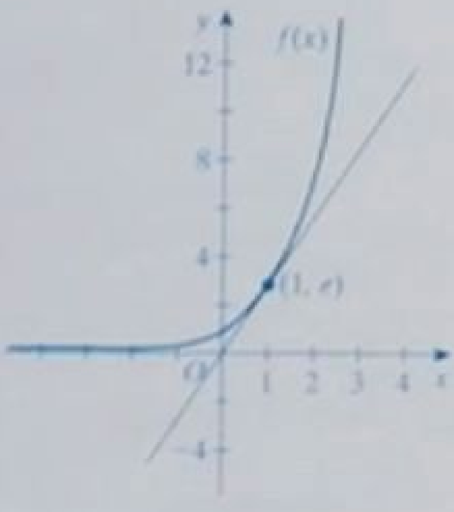
$$f'\left(\frac{3}{4}\pi\right) = -\frac{\sqrt{2}}{2}$$

The equation of the tangent line is:

$$y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{3}{4}\pi\right)$$

$$y = -\frac{\sqrt{2}}{2}\left(x - \frac{3}{4}\pi - 1\right)$$

(4) $y = e^x$ $\qquad\qquad (1, e)$



[Sol] Letting $\quad f(x) = e^x,$

$$f'(x) = e^x$$

$$f'(1) = e$$

The equation of the tangent line is:

$$y - e = e(x - 1)$$

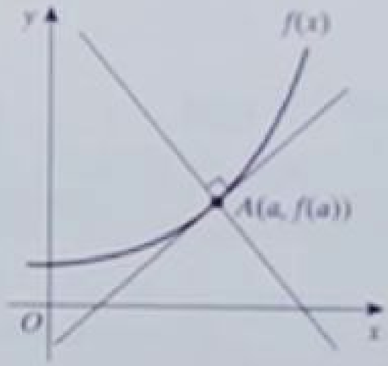
$$y = ex$$

Concavity and Tangent Lines

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	1	2~

Given a curve with point A and its tangent line at point A, the *normal line* at point A is the line that passes through point A and is perpendicular to the tangent line at point A.



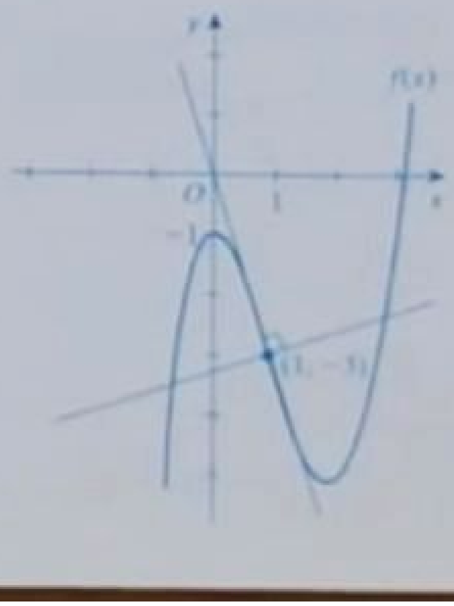
1. Given a curve with point $B(b, f(b))$, find the equation of the normal line at point B.
   [Sol] The slope of the tangent line is: $\boxed{f'(b)}$
   Since the normal line is perpendicular to the tangent line,
   the slope of the normal line is: $\boxed{-\dfrac{1}{f'(b)}}$
   Therefore, the equation of the normal line is:
   $$y - f(b) = -\frac{1}{f'(b)}(x - b)$$

2. Find the equations of the tangent line and the normal line to each of the following curves at the given point. Then, graph the curve, the tangent line and the normal line.

(1) $y = x^3 - 3x^2 - 1$ $\quad(1, -3)$

[Sol] Letting $f(x) = x^3 - 3x^2 - 1$,
$f'(x) = 3x^2 - 6x$
$f'(1) = -3$

The equation of the tangent line is:
$$y = -3x$$
The equation of the normal line is:
$$y = \frac{1}{3}x - \frac{10}{3}$$



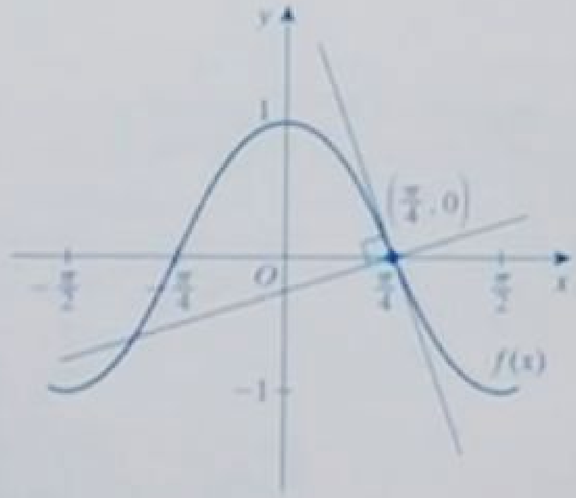
O 38 b

(2) $y = \cos 2x$ $\quad\quad\left(\frac{\pi}{4}, 0\right)$

[Sol] Letting $f(x) = \cos 2x,$

$$f'(x) = -2\sin 2x$$

$$f'\left(\frac{\pi}{4}\right) = -2$$



The equation of the tangent line is:

$$y = -2\left(x - \frac{\pi}{4}\right)$$

The equation of the normal line is:

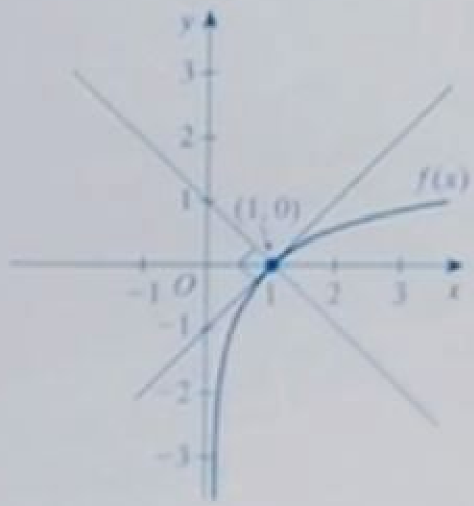
$$y = \frac{1}{2}\left(x - \frac{\pi}{4}\right)$$

(3) $y = \ln x$ $\quad\quad(1, 0)$

[Sol] Letting $f(x) = \ln x,$

$$f'(x) = \frac{1}{x}$$

$$f'(1) = 1$$



The equation of the tangent line is:

$$y = x - 1$$

The equation of the normal line is:

$$y = -x + 1$$

Concavity and Tangent Lines

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2–

Find the equations of the tangent line and the normal line to each of the following curves at the given point. Then, graph the curve, the tangent line and the normal line.

(1) $y = \tan x$ $\qquad (\pi, 0)$

[Sol] Letting $f(x) = \tan x$,

$$f'(x) = \frac{1}{\cos^2 x}$$

$$f'(\pi) = 1$$

The equation of the tangent line is:

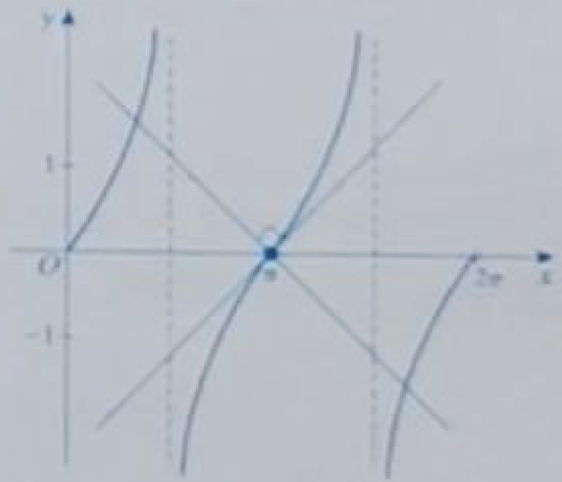
$$y - 0 = 1(x - \pi)$$

$$y = x - \pi$$

The equation of the normal line is:

$$y - 0 = -1(x - \pi)$$

$$y = -x + \pi$$



(2) $y = \dfrac{2}{1-x}$ $\qquad (0, 2)$

[Sol] Letting $f(x) = \dfrac{2}{1-x}$,

$$f'(x) = \frac{2}{(1-x)^2}$$

$$f'(0) = 2$$

The equation of the tangent line is:

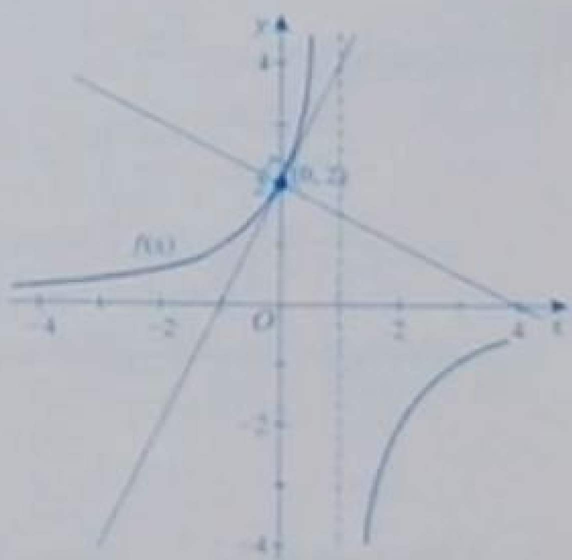
$$y - 2 = 2(x - 0)$$

$$y = 2x + 2$$

The equation of the normal line is:

$$y - 2 = -\frac{1}{2}(x - 0)$$

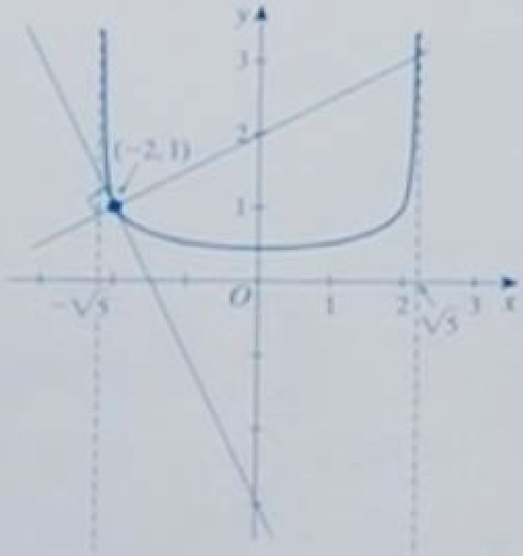
$$y = -\frac{1}{2}x + 2$$



O 39 b

(3) $y=\dfrac{1}{\sqrt{5-x^2}}$ $(-2, 1)$

[Sol] Letting $f(x)=\dfrac{1}{\sqrt{5-x^2}}$,



$$f'(x)=\frac{x}{(5-x^2)\sqrt{5-x^2}}$$

$$f'(-2)=-2$$

The equation of the tangent line is:

$$y-1=-2(x+2)$$

$$y=-2x-3$$

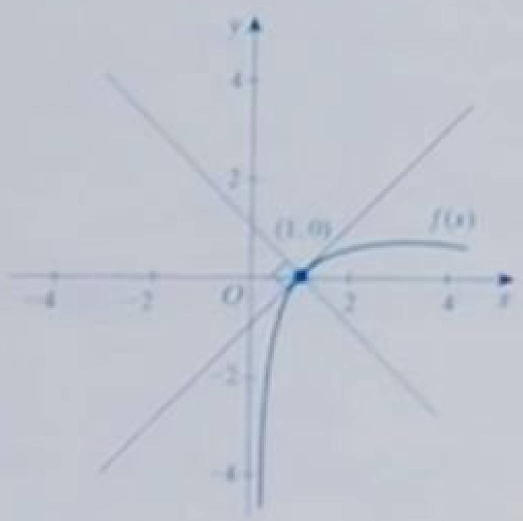
The equation of the normal line is:

$$y-1=\frac{1}{2}(x+2)$$

$$y=\frac{1}{2}(x+4)$$

(4) $y=\dfrac{\ln x}{x}$ $(1, 0)$

[Sol] Letting $f(x)=\dfrac{\ln x}{x}$,



$$f'(x)=\frac{1-\ln x}{x^2}$$

$$f'(1)=1$$

The equation of the tangent line is:

$$y-0=1(x-1)$$

$$y=x-1$$

The equation of the normal line is:

$$y-0=-1(x-1)$$

$$y=-x+1$$

KUMON

O 40 a

Concavity and Tangent Lines

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

1. For the given function, create a variation table indicating where the curve is concave up and concave down, and note the point(s) of inflection. Then, state the asymptote(s), (if any), and draw the graph.

$$y = \frac{x^3 + x^2 + 1}{x^2 + 1}$$

[Sol] $y = \dfrac{x^3 + x^2 + 1}{x^2 + 1} = x + 1 - \dfrac{x}{x^2 + 1}$ $\cdots$ ①

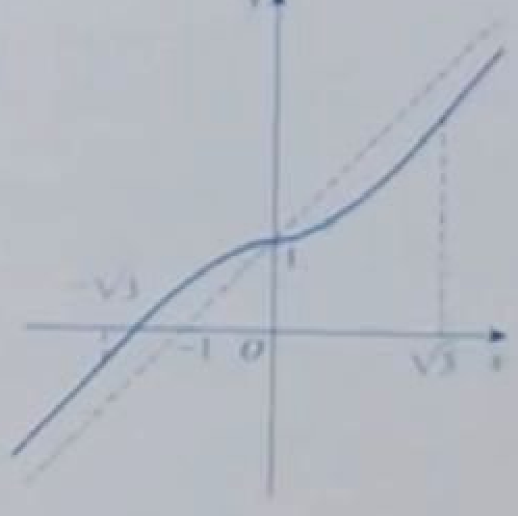
$$y' = 1 - \frac{1 - x^2}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}$$

$$y'' = \frac{2x(3 - x^2)}{(x^2 + 1)^3} = -\frac{2x(x + \sqrt{3})(x - \sqrt{3})}{(x^2 + 1)^3}$$

x	$\cdots$	$-\sqrt{3}$	$\cdots$	0	$\cdots$	$\sqrt{3}$	$\cdots$
y'	+	+	+	0	+	+	+
y''	+	0	−	0	+	0	−
y	$\smile$	i.p.	$\frown$	i.p., 1	$\smile$	i.p.	$\frown$

From ①,

The asymptote is: $y = x + 1$



O 40 b

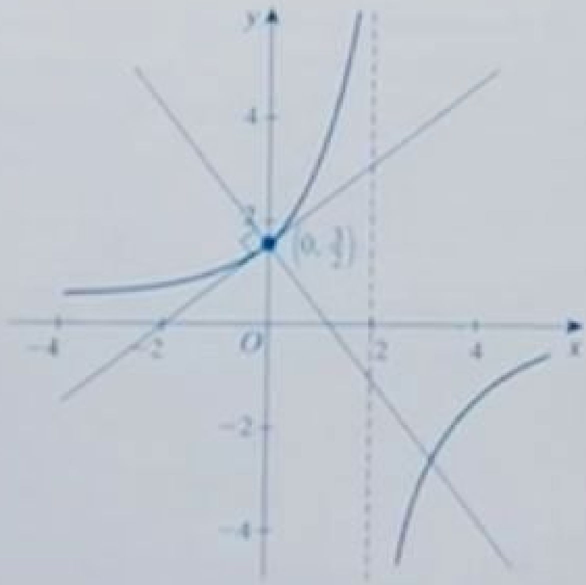
2. Find the equation of the tangent line to $y = \dfrac{3}{2-x}$ at $\left(0, \dfrac{3}{2}\right)$.

Then, graph the curve, the tangent line, and the normal line.

[Sol] Letting $f(x) = \dfrac{3}{2-x}$,

$$f'(x) = \frac{3}{(2-x)^2}$$

$$f'(0) = \frac{3}{4}$$



The equation of the tangent line is:

$$y - \frac{3}{2} = \frac{3}{4}(x-0)$$

$$y = \frac{3}{4}(x+2)$$

The equation of the normal line is:

$$y - \frac{3}{2} = -\frac{4}{3}(x-0)$$

$$y = -\frac{4}{3}x + \frac{3}{2}$$

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

In each exercise, obtain the maximum and minimum values of the given function.

Ex. $f(x) = \dfrac{x}{x^2+4} \qquad (-3 \leq x \leq 3)$

[Sol] $f'(x) = \dfrac{4-x^2}{(x^2+4)^2}$

$f'(x) = 0$ when $x = \pm 2$

x	-3	$\cdots$	-2	$\cdots$	2	$\cdots$	3
$f'(x)$	$-$	$-$	0	$+$	0	$-$	$-$
$f(x)$	$-\frac{3}{13}$	$\searrow$	$-\frac{1}{4}$	$\nearrow$	$\frac{1}{4}$	$\searrow$	$\frac{3}{13}$

From the table,

The maximum value is $\dfrac{1}{4}$, (at $x = 2$).

The minimum value is $-\dfrac{1}{4}$, (at $x = -2$).

(1) $f(x) = \dfrac{x}{x^2+4} \qquad (-1 \leq x \leq 1)$

[Sol] $f'(x) = \dfrac{4-x^2}{(x^2+4)^2}$

$f'(x) = 0$ when $x = \pm 2$

x	-1	$\cdots$	1
$f'(x)$	$+$	$+$	$+$
$f(x)$	$-\frac{1}{5}$	$\nearrow$	$\frac{1}{5}$

From the table,

The maximum value is $\dfrac{1}{5}$, (at $x = 1$).

The minimum value is $-\dfrac{1}{5}$, (at $x = -1$).

O 41 b

(2) $f(x)=\dfrac{2(x-1)}{x^2-2x+2}$

[Sol] $f'(x)=-\dfrac{2x(x-2)}{(x^2-2x+2)^2}$

$f'(x)=0$ when $x=\boxed{0},\boxed{2}$ And, $\lim\limits_{x\to\pm\infty} f(x)=\boxed{0}$

x	$-\infty$	$\cdots$	0	$\cdots$	2	$\cdots$	$+\infty$
$f'(x)$	$-$	$-$	0	$+$	0	$-$	$-$
$f(x)$	0	$\searrow$	-1	$\nearrow$	1	$\searrow$	0

From the table,

The maximum value is 1, (at $x=2$).

The minimum value is -1, (at $x=0$).

(3) $f(x)=\dfrac{1-x}{x^2+2}$

[Sol] $f'(x)=\dfrac{x^2-2x-2}{(x^2+2)^2}$

$f'(x)=0$ when $x=1\pm\sqrt{3}$ And, $\lim\limits_{x\to\pm\infty} f(x)=0$

x	$-\infty$	$\cdots$	$1-\sqrt{3}$	$\cdots$	$1+\sqrt{3}$	$\cdots$	$+\infty$
$f'(x)$	$+$	$+$	0	$-$	0	$+$	$+$
$f(x)$	0	$\nearrow$	$\frac{\sqrt{3}+1}{4}$	$\searrow$	$-\frac{\sqrt{3}-1}{4}$	$\nearrow$	0

From the table,

The maximum value is $\dfrac{\sqrt{3}+1}{4}$, (at $x=1-\sqrt{3}$).

The minimum value is $-\dfrac{\sqrt{3}-1}{4}$, (at $x=1+\sqrt{3}$).

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

In each exercise, obtain the maximum and minimum values of the given function.

(1) $y = x - \sin x \qquad (0 \leq x \leq 2\pi)$

[Sol] $y' = 1 - \cos x \geq 0$

x	0	$\cdots$	2π
y'	0	$+$	0
y	0	$\nearrow$	2π

From the table,

The maximum value is 2π, (at $x = 2\pi$).

The minimum value is 0, (at $x = 0$).

(2) $y = 2x - \cos 2x \qquad (0 \leq x \leq 2\pi)$

[Sol] $y' = 2 + 2\sin 2x \qquad (0 \leq 2x \leq 4\pi)$

$y' = 0$ when $2x = \frac{3\pi}{2}, \frac{7\pi}{2}$, which means when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

x	0	$\cdots$	$\frac{3\pi}{4}$	$\cdots$	$\frac{7\pi}{4}$	$\cdots$	2π
y'	$+$	$+$	0	$+$	0	$+$	$+$
y	-1	$\nearrow$	$\frac{3\pi}{2}$	$\nearrow$	$\frac{7\pi}{2}$	$\nearrow$	$4\pi - 1$

From the table,

The maximum value is $4\pi - 1$, (at $x = 2\pi$).

The minimum value is -1, (at $x = 0$).

O 42 b

(3) $y = \sin x(1 - \sin x)$ $(-\pi \le x \le \pi)$

[Sol] $y' = \cos x(1 - 2\sin x)$

$y' = 0$ when $x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

x	$-\pi$	$\cdots$	$-\frac{\pi}{2}$	$\cdots$	$\frac{\pi}{6}$	$\cdots$	$\frac{\pi}{2}$	$\cdots$	$\frac{5\pi}{6}$	$\cdots$	π
y'	$-$	$-$	0	$+$	0	$-$	0	$+$	0	$-$	$-$
y	0	$\searrow$	-2	$\nearrow$	$\frac{1}{4}$	$\searrow$	0	$\nearrow$	$\frac{1}{4}$	$\searrow$	0

From the table,

The maximum value is $\frac{1}{4}$, $\left(\text{at } x = \frac{\pi}{6}, \frac{5\pi}{6}\right)$.

The minimum value is -2, $\left(\text{at } x = -\frac{\pi}{2}\right)$.

(4) $y = \sin 2x$ $(-\pi \le x \le \pi)$

[Sol] $y' = 2\cos 2x$ $(-2\pi \le 2x \le 2\pi)$

$y' = 0$ when $2x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$,

which means when $x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

x	$-\pi$	$\cdots$	$-\frac{3\pi}{4}$	$\cdots$	$-\frac{\pi}{4}$	$\cdots$	$\frac{\pi}{4}$	$\cdots$	$\frac{3\pi}{4}$	$\cdots$	π
y'	$+$	$+$	0	$-$	0	$+$	0	$-$	0	$+$	$+$
y	0	$\nearrow$	1	$\searrow$	-1	$\nearrow$	1	$\searrow$	-1	$\nearrow$	0

From the table,

The maximum value is 1, $\left(\text{at } x = -\frac{3\pi}{4}, \frac{\pi}{4}\right)$.

The minimum value is -1, $\left(\text{at } x = -\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

O 43 a Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

In each exercise, obtain the maximum and minimum values of the given function.

(1) $y = xe^{-x^2}$ $(0 \leq x \leq 1)$

[Sol] $y' = (1 - 2x^2)e^{-x^2}$

$y' = 0$ when $x = \dfrac{\sqrt{2}}{2}$ $(0 \leq x \leq 1)$

x	0	$\cdots$	$\frac{\sqrt{2}}{2}$	$\cdots$	1
y'	+	+	0	–	–
y	0	$\nearrow$	$\frac{1}{\sqrt{2e}}$	$\searrow$	e^{-1}

From the table,

The maximum value is $\dfrac{1}{\sqrt{2e}}$, $\left(\text{at } x = \dfrac{\sqrt{2}}{2}\right)$.

The minimum value is 0, (at $x = 0$).

(2) $y = e^{-x} - e^x$ $(-1 \leq x \leq 2)$

[Sol] $y' = -e^{-x} - e^x < 0$

x	-1	$\cdots$	2
y'	–	–	–
y	$e - e^{-1}$	$\searrow$	$e^{-2} - e^2$

From the table,

The maximum value is $e - e^{-1}$, (at $x = -1$).

The minimum value is $e^{-2} - e^2$, (at $x = 2$).

O 43 b

(3) $y = x\ln x$

[Sol] $y' = \ln x + 1$

$y' = 0$ when $x = \frac{1}{e}$ And, $\lim_{x \to +\infty} y = +\infty$

x	0	$\cdots$	$\frac{1}{e}$	$\cdots$	$+\infty$
y'		$-$	0	$+$	$+$
y		$\searrow$	$-\frac{1}{e}$	$\nearrow$	$+\infty$

From the table,

There is no maximum value.

The minimum value is $-\frac{1}{e}$, $\left(\text{at } x = \frac{1}{e}\right)$.

(4) $y = \frac{\ln x}{x^2}$ $(1 \le x \le e)$

[Sol] $y' = \frac{1 - 2\ln x}{x^3}$

$y' = 0$ when $x = \sqrt{e}$

x	1	$\cdots$	$\sqrt{e}$	$\cdots$	e
y'	$+$	$+$	0	$-$	$-$
y	0	$\nearrow$	$\frac{1}{2e}$	$\searrow$	$\frac{1}{e^2}$

From the table,

The maximum value is $\frac{1}{2e}$, (at $x = \sqrt{e}$).

The minimum value is 0, (at $x = 1$).

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

In each exercise, obtain the maximum and minimum values of the given function.

(1) $y = x - 2 + \sqrt{4 - x^2}$

[Sol] Since $4 - x^2 \geq 0$, the domain is $\boxed{-2} \leq x \leq \boxed{2}$.

Over this domain,

$$y' = 1 + \frac{-2x}{2\sqrt{4 - x^2}} = \frac{\sqrt{4 - x^2} - x}{\sqrt{4 - x^2}}$$

$y' = 0$ when $\sqrt{4 - x^2} = x$, $(x \geq 0)$, which means when $x = \sqrt{2}$

x	-2	$\cdots$	$\sqrt{2}$	$\cdots$	2
y'		$+$	0	$-$	
y	-4	$\nearrow$	$2(\sqrt{2} - 1)$	$\searrow$	0

From the table,

The maximum value is $2(\sqrt{2} - 1)$, (at $x = \sqrt{2}$).

The minimum value is -4, (at $x = -2$).

(2) $y = x^3 \sqrt{4 - x^2}$

[Sol] Since $4 - x^2 \geq 0$, the domain is $-2 \leq x \leq 2$.

$$y' = \frac{-4x^2(x^2 - 3)}{\sqrt{4 - x^2}}$$

$y' = 0$ when $x = 0$ and $\pm\sqrt{3}$

x	-2	$\cdots$	$-\sqrt{3}$	$\cdots$	0	$\cdots$	$\sqrt{3}$	$\cdots$	2
y'		$-$	0	$+$	0	$+$	0	$-$	
y	0	$\searrow$	$-3\sqrt{3}$	$\nearrow$	0	$\nearrow$	$3\sqrt{3}$	$\searrow$	0

From the table,

The maximum value is $3\sqrt{3}$, (at $x = \sqrt{3}$).

The minimum value is $-3\sqrt{3}$, (at $x = -\sqrt{3}$).

O 44 b

(3) $y = \sqrt{x-1} + \sqrt{2-x}$

[Sol] $y' = \dfrac{\sqrt{2-x} - \sqrt{x-1}}{2\sqrt{x-1}\sqrt{2-x}} = \dfrac{3-2x}{2\sqrt{x-1}\sqrt{2-x}(\sqrt{2-x}+\sqrt{x-1})}$

Over the domain $1 \le x \le 2$,

$y' = 0$ when $x = \dfrac{3}{2}$

x	1	$\cdots$	$\frac{3}{2}$	$\cdots$	2
y'		+	0	−	
y	1	↗	$\sqrt{2}$	↘	1

From the table,

The maximum value is $\sqrt{2}$. $\left(\text{at } x = \dfrac{3}{2}\right)$.

The minimum value is 1. (at $x = 1, 2$).

(4) $y = x + \sqrt{2x - x^2}$

[Sol] Since $2x - x^2 \ge 0$, the domain is $0 \le x \le 2$.

$y' = \dfrac{\sqrt{2x-x^2} - (x-1)}{\sqrt{2x-x^2}}$

$y' = 0$ when $\sqrt{2x - x^2} = x - 1$, $(x \ge 1)$, which means when $x = \dfrac{2+\sqrt{2}}{2}$

x	0	$\cdots$	$\frac{2+\sqrt{2}}{2}$	$\cdots$	2
y'		+	0	−	
y	0	↗	$1+\sqrt{2}$	↘	2

From the table,

The maximum value is $1+\sqrt{2}$. $\left(\text{at } x = \dfrac{2+\sqrt{2}}{2}\right)$

The minimum value is 0. (at $x = 0$)

KUMON

O 45 a

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

1. Obtain the maximum value of the given function by applying the following steps.

$y = \sin^3 x + \cos^3 x - 3\sin x\cos x$

(1) Letting $t = \sin x + \cos x$, express y in terms of t.

[Sol] $t^2 = \sin^2 x + \boxed{\cos^2 x} + 2\boxed{\sin x\cos x} = \boxed{1} + 2\boxed{\sin x\cos x}$

Therefore, $\sin x\cos x = \boxed{\frac{1}{2}}(t^2 - \boxed{1})$

$$y = (\sin x + \cos x)^3 - 3\sin x\cos x(\sin x + \cos x) - 3\sin x\cos x$$
$$= t^3 - \frac{3}{2}t(t^2-1) - \frac{3}{2}(t^2-1)$$
$$= -\frac{1}{2}(t^3 + 3t^2 - 3t - 3)$$

(2) Obtain the range of values of t.

[Sol] Since $t = \sqrt{2}\sin\left(x + \boxed{\frac{\pi}{4}}\right)$, $\boxed{-\sqrt{2}} \le t \le \boxed{\sqrt{2}}$

(3) Obtain the maximum value of y.

[Sol] From (1),

$$y' = -\frac{3}{2}(t^2 + 2t - 1)$$
$$= -\frac{3}{2}[t-(-1+\sqrt{2})][t-(-1-\sqrt{2})]$$

Over the range found in (2), $y' = 0$ when $t = \boxed{-1+\sqrt{2}}$

t	$-\sqrt{2}$	$\cdots$	$-1+\sqrt{2}$	$\cdots$	$\sqrt{2}$
y'	+	+	0	–	–
y	$-\frac{3+\sqrt{2}}{2}$	↗	$2\sqrt{2}-1$	↘	$-\frac{3-\sqrt{2}}{2}$

From the table,

The maximum value is: $2\sqrt{2}-1$, (at $t = -1+\sqrt{2}$)

O 45 b

2. Obtain the minimum value of the given function by applying the following steps.

$$y = \frac{x^4 + x^2 + 1}{x^3 + x} \qquad \text{(where } x > 0\text{)}$$

(1) Letting $t = x + \frac{1}{x}$, express y in terms of t.

[Sol] Dividing each term of the RHS of y by x^2,

$$y = \frac{x^2 + 1 + \frac{1}{x^2}}{x + \frac{1}{x}} \qquad \cdots ①$$

Substituting $t = x + \frac{1}{x}$ into ①,

$$y = \frac{t^2 - 1}{t} = t - \frac{1}{t}$$

(2) Obtain the range of values of t.

[Sol] Since $t = x + \frac{1}{x}$,

Since the arithmetic mean $\geqq$ the geometric mean, (From J197b)

$$t = x + \frac{1}{x} \geqq 2$$

(3) Obtain the minimum value of y.

[Sol] $y' = 1 + \frac{1}{t^2}$

Since $1 + \frac{1}{t^2} > 0$, y is an increasing function.

Therefore, y has a minimum value when $t = 2$.

The minimum value is: $\frac{3}{2}$, (at $t = 2$)

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Obtain the maximum and minimum values of the given function by applying the following steps.

$$y = \frac{2x\sqrt{1-x^2}}{x+\sqrt{1-x^2}+2} \quad (\text{where } -1 \leq x \leq 1)$$

(1) Letting $t = x + \sqrt{1-x^2} + 2$, express y in terms of t.

[Sol] The denominator of y is t.

Expressing the numerator, $2x\sqrt{1-x^2}$, in terms of t,

From $t = x + \sqrt{1-x^2} + 2$,

$t - 2 = x + \sqrt{1-x^2}$

Squaring both sides,

$(t-2)^2 = x^2 + 2x\sqrt{1-x^2} + 1 - x^2$

$2x\sqrt{1-x^2} = (t-2)^2 - 1$

Therefore, $y = \dfrac{t^2 - 4t + 3}{t}$

(2) Obtain the range of values of t.

[Sol] From (1), $t = x + \sqrt{1-x^2} + 2$

$$t' = 1 - \frac{x}{\sqrt{1-x^2}} = \frac{\sqrt{1-x^2} - x}{\sqrt{1-x^2}} \qquad (-1 \leq x \leq 1)$$

$t' = 0$ when $\sqrt{1-x^2} - x = 0$

$\sqrt{1-x^2} = x$

$x = \dfrac{1}{\sqrt{2}}$ $\left(x = -\dfrac{1}{\sqrt{2}} \text{ is an extraneous solution.}\right)$

x	-1	$\cdots$	$\frac{1}{\sqrt{2}}$	$\cdots$	1
t'		$+$	0	$-$	
t	1	↗	$2+\sqrt{2}$	↘	3

From the table, $1 \leq t \leq 2 + \sqrt{2}$

O 46 b

(3) Obtain the maximum and minimum values of y.

[Sol] From (1),

$$y = \frac{t^2 - 4t + 3}{t}$$

$$y' = \frac{t^2 - 3}{t^2}$$

Over the range found in (2),

$y' = 0$ when $t = \sqrt{3}$

t	1	$\cdots$	$\sqrt{3}$	$\cdots$	$2+\sqrt{2}$
y'	$-$	$-$	0	+	+
y	0	$\searrow$	$2\sqrt{3}-4$	$\nearrow$	$\frac{2-\sqrt{2}}{2}$

From the table,

The maximum value is $\dfrac{2-\sqrt{2}}{2}$. (at $t = 2 + \sqrt{2}$)

The minimum value is $2\sqrt{3} - 4$. (at $t = \sqrt{3}$)

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	1	2~

1. Given that x and y are positive real numbers such that $x^2-2x+4y^2=0$, obtain the maximum value of the product xy, by applying the following steps.

(1) Express y in terms of x, and state the domain of x.

[Sol] From $x^2-2x+4y^2=0$,

$$y=\frac{1}{2}\sqrt{2x-x^2},\qquad 0<x<2$$

(2) Express xy in terms of x.

[Sol] From (1), since $y=\frac{1}{2}\sqrt{2x-x^2}$,

$$xy=\frac{1}{2}x\sqrt{2x-x^2}$$

(3) Letting $f(x)=xy$, obtain the maximum value of $f(x)$.

[Sol] From (2), $f(x)=\frac{1}{2}x\sqrt{2x-x^2}$

$$f'(x)=\frac{1}{2}\left(\sqrt{2x-x^2}+x\cdot\frac{2-2x}{2\sqrt{2x-x^2}}\right)=\frac{-x\left(x-\frac{3}{2}\right)}{\sqrt{2x-x^2}}$$

x	0	$\cdots$	$\frac{3}{2}$	$\cdots$	2
$f'(x)$	/	+	0	−	/
$f(x)$	/	↗	$\frac{3\sqrt{3}}{8}$	↘	/

Over the domain stated in (1),

$f'(x)=0$ when $x=\frac{3}{2}$. At this value, $y=\frac{\sqrt{3}}{4}$

Therefore, the maximum value of $f(x)$ is $\frac{3\sqrt{3}}{8}$

O 47 b

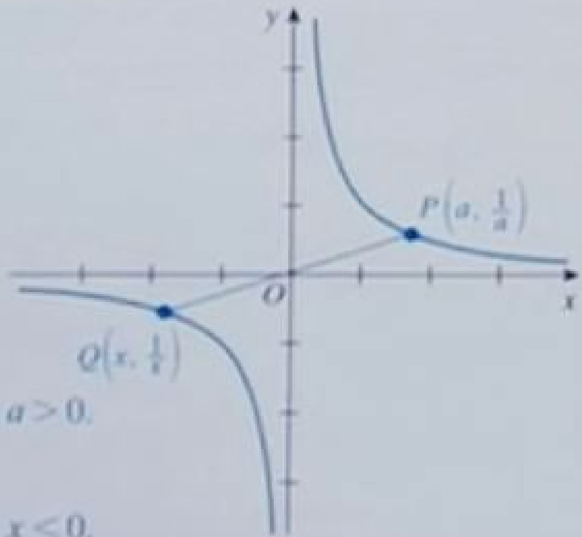
2. Given the curve $xy=1$, let $P\left(a, \frac{1}{a}\right)$ be a fixed point on the curve in the 1^{st} Quadrant, and let Q be a moving point on the curve in the 3^{rd} Quadrant. Obtain the coordinates of point Q, at which the length of line segment PQ is at a minimum value.

[Sol] Letting the coordinates of Q be $\left(x, \frac{1}{x}\right)$,

$$f(x)=PQ^2=(x-a)^2+\left(\frac{1}{x}-\frac{1}{a}\right)^2$$

$$f'(x)=2(x-a)+2\left(\frac{1}{x}-\frac{1}{a}\right)\left(-\frac{1}{x^2}\right)$$

$$=2(x-a)\left(1+\frac{1}{ax^3}\right)$$



Since $P\left(a, \frac{1}{a}\right)$ is in the 1^{st} Quadrant, $a>0$.

Since $Q\left(x, \frac{1}{x}\right)$ is in the 3^{rd} Quadrant, $x<0$.

x	$\cdots$	$-\frac{1}{\sqrt[3]{a}}$	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	minimum	$\nearrow$

From the table,

$f(x)$ reaches a minimum when $x=-\frac{1}{\sqrt[3]{a}}$.

Therefore, the coordinates of Q are $\left(-\frac{1}{\sqrt[3]{a}}, -\sqrt[3]{a}\right)$.

KUMON

O 48 a

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

1. Let $P\left(t, \frac{t^2}{2}\right)$ be a point on the curve of the parabola $y = \frac{x^2}{2}$. Let Q be the point on the curve which is intersected by the normal line to P. Complete the following exercises.

(1) Determine the equation of the normal line, PQ.

[Sol] When P has coordinates $(0, 0)$, Q does not exist. Therefore, let $t \neq 0$.

From $P\left(t, \frac{t^2}{2}\right)$, the slope of the tangent line to $y = \frac{x^2}{2}$ is t

Therefore, the equation of the normal line, PQ, can be expressed as:

$$y = -\frac{1}{t}(x - t) + \frac{t^2}{2}$$

(2) Obtain the coordinates of point Q.

[Sol] Q has x-coordinate: $\frac{x^2}{2} = -\frac{1}{t}(x - t) + \frac{t^2}{2}$

This is the solution of the equation $(x - t)(tx + t^2 + 2) = 0$.

Since $x \neq t$, $\quad x = -\left(t + \frac{2}{t}\right)$

Therefore, Q has coordinates: $\left(-\left(t + \frac{2}{t}\right), \ \frac{1}{2}\left(t + \frac{2}{t}\right)^2\right)$

(3) Find the minimum value of the length of line segment PQ, as P moves along the curve.

[Sol] $PQ^2 = \left(2t + \frac{2}{t}\right)^2 + \left[\frac{1}{2}\left(t + \frac{2}{t}\right)^2 - \frac{t^2}{2}\right]^2 = 4\left(t^2 + \frac{3}{t^2} + \frac{1}{t^4} + 3\right)$

Letting $PQ^2 = f(t)$, $\quad f'(t) = 8\left(t - \frac{3}{t^3} - \frac{2}{t^5}\right) = \frac{8(t^2 - 2)(t^2 + 1)^2}{t^5}$

t	$\cdots$	$-\sqrt{2}$	$\cdots$	0	$\cdots$	$\sqrt{2}$	$\cdots$
$f'(t)$	$-$	0	$+$		$-$	0	$+$
$f(t)$	$\searrow$	27	$\nearrow$		$\searrow$	27	$\nearrow$

From the table, $f(t)$ has a minimum value of 27, at $t = \pm\sqrt{2}$.

Since $\sqrt{27} = 3\sqrt{3}$

PQ has a minimum value of $3\sqrt{3}$

O 48 b

2. Let $P(t, s)$ (where $t > 0$) be a point on the curve of the parabola $y = 1 - x^2$. Obtain the minimum value of the area of the triangle formed by the x and y axes and the line tangent to the curve at point P.

[Sol] From $y = 1 - x^2$,

$$y' = -2x$$

Point P becomes $(t, 1 - t^2)$, and since $t > 0$,

The equation of the tangent line can be expressed as:

$$y = -2t(x - t) + 1 - t^2$$
$$y = -2tx + 1 + t^2 \qquad \cdots ①$$

Line ① intersects the x and y axes at points: $\left(\dfrac{1 + t^2}{2t}, 0\right)$, $(0, 1 + t^2)$

Letting S be the area of the triangle,

$$S = \frac{1}{2} \cdot \frac{1 + t^2}{2t} \cdot (1 + t^2) = \frac{(1 + t^2)^2}{4t}$$

$$S' = \frac{1}{4} \cdot \frac{(t^2 + 1)(3t^2 - 1)}{t^2}$$

t	$\cdots$	$\frac{1}{\sqrt{3}}$	$\cdots$
S'	$-$	0	$+$
S	$\searrow$	$\frac{4\sqrt{3}}{9}$	$\nearrow$

From the table,

The minimum value of the area is $\dfrac{4\sqrt{3}}{9}$.

KUMON

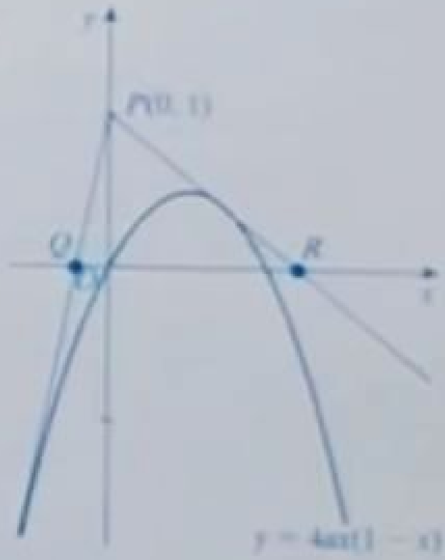
O 49 a — Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	—	—	—	—

1. Given the parabola $y = 4ax(1 - x)$ (where $0 < a < 1$) and point $P(0, 1)$, draw two lines from P which are tangent to the parabola and which intersect the x-axis at points Q and R.

(1) Determine the equations of the two lines.



[Sol] From $y = 4ax(1 - x)$

$y' = -4a(2x - 1)$

The general equation of a line tangent at $x = t$ is:

$y = -4a(2t - 1)(x - t) + 4at(1 - t)$

$y = -4a(2t - 1)x + 4at^2$ …①

Since ① passes through point $P(0, 1)$,

$4at^2 = 1$ and $t = \pm\dfrac{1}{2\sqrt{a}}$

Substituting these values into ①, the equations of the two lines are:

$$\begin{cases} y = 4(a + \sqrt{a})x + 1 \\ y = 4(a - \sqrt{a})x + 1 \end{cases} \quad \text{…②}$$

(2) Obtain the value of a at which the area of $\triangle PQR$ has a minimum value.

[Sol] From ②, letting $y = 0$,

Q has x-coordinate $x = -\dfrac{1}{4(a + \sqrt{a})}$, and R has x-coordinate $x = -\dfrac{1}{4(a - \sqrt{a})}$

Therefore, $QR = -\dfrac{1}{4(a - \sqrt{a})} + \dfrac{1}{4(a + \sqrt{a})} = \dfrac{1}{2\sqrt{a}(1 - a)}$

Letting S be the area of $\triangle PQR$, $S = \dfrac{1}{4\sqrt{a}(1 - a)}$

In order to find the minimum value of S, letting $t = \sqrt{a}$, we need to determine the maximum value of $\sqrt{a}(1 - a) = t(1 - t^2) = f(t)$ (where $0 < t < 1$).

$f'(t) = 1 - 3t^2 = -3\left(t + \dfrac{1}{\sqrt{3}}\right)\left(t - \dfrac{1}{\sqrt{3}}\right)$

t	0	…	$\frac{1}{\sqrt{3}}$	…	1
$f'(t)$		+	0	−	
$f(t)$	0	↗	maximum	↘	0

From the table, $f(t)$ has a maximum value when $t = \dfrac{1}{\sqrt{3}}$

Since $t = \sqrt{a}$, $\sqrt{a} = \dfrac{1}{\sqrt{3}}$

Therefore, the area S of $\triangle PQR$ has a minimum value when $a = \dfrac{1}{3}$

2. Given the curve $y = e^x$, let A and B be the points on the curve at $x = 0$ and $x = 1$ respectively. Placing point P on curve y, somewhere between A and B, obtain the maximum value of the area of $\triangle PAB$.

[Sol] Letting the coordinates of P be (x, e^x), and letting $S(x)$ be the area of $\triangle PAB$,

$$S(x) = \frac{e-1}{2} - \frac{e^x-1}{2} - \frac{(e-1)(1-x)}{2}$$
$$= \frac{1}{2}(1 - x + ex - e^x) \qquad (0 < x < 1)$$

Alternate Solution for the calculation of $S(x)$:

$$S(x) = \frac{1+e}{2} - \frac{(1+e^x)x}{2} - \frac{(e+e^x)(1-x)}{2}$$
$$= \frac{1}{2}(1 - x + ex - e^x) \qquad (0 < x < 1)$$

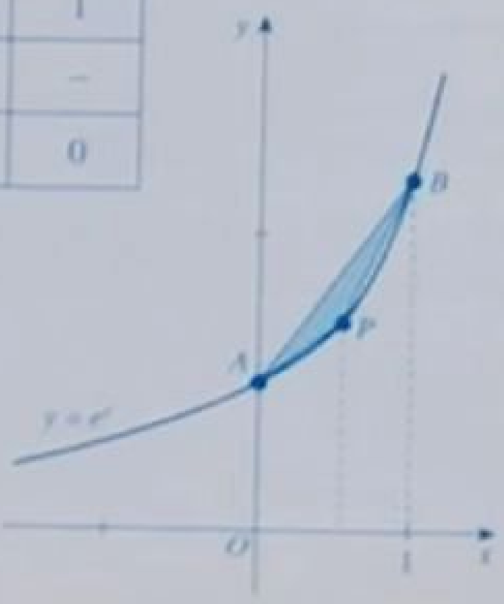
For a review of finding the area of trapezoids, refer to L 141a.

$$S'(x) = -\frac{1}{2}[e^x - (e-1)]$$

When $S'(x) = 0$, $e^x = e - 1$

Therefore, $x = \ln(e-1)$

x	0	···	$\ln(e-1)$	···	1
$S'(x)$	+	+	0	−	−
$S(x)$	0	↗	maximum	↘	0



$S(x)$ has a maximum value when $x = \ln(e-1)$.

Therefore, the maximum value of the area of $\triangle PAB$ is:

$$\frac{1}{2}[2 - e + (e-1)\ln(e-1)]$$

Maxima and Minima

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

1. In each exercise, obtain the maximum and minimum values of the given function.

(1) $y = x - \cos x \qquad (0 \leq x \leq 2\pi)$

[Sol] $y' = 1 + \sin x \geq 0$

x	0	$\cdots$	$\frac{3\pi}{2}$	$\cdots$	2π
y'	$+$	$+$	0	$+$	$+$
y	-1	$\nearrow$	$\frac{3\pi}{2}$	$\nearrow$	$2\pi - 1$

From the table,

The maximum value is $2\pi - 1$, (at $x = 2\pi$).

The minimum value is -1, (at $x = 0$).

(2) $y = \sqrt{1+x} + \sqrt{1-x}$

[Sol] The domain is $-1 \leq x \leq 1$.

$$y' = \frac{\sqrt{1-x} - \sqrt{1+x}}{2\sqrt{1-x^2}}$$

$y' = 0$ when $\sqrt{1-x} = \sqrt{1+x}$, which means when $x = 0$

x	-1	$\cdots$	0	$\cdots$	1
y'		$+$	0	$-$	
y	$\sqrt{2}$	$\nearrow$	2	$\searrow$	$\sqrt{2}$

From the table,

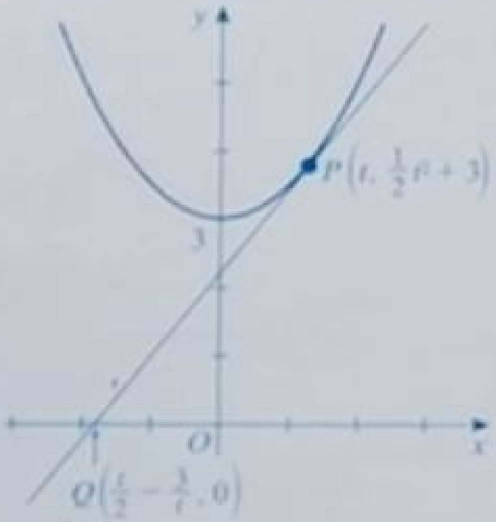
The maximum value is 2, (at $x = 0$).

The minimum value is $\sqrt{2}$, (at $x = \pm 1$).

O 50 b

2. Given the parabola $y=\frac{1}{2}x^2+3$, letting $P\left(t,\ \frac{1}{2}t^2+3\right)$ be a point on the curve, extend the line tangent to the curve at P down to the x-axis, and denote the point of intersection with the x-axis as Q. Then, obtain the value of t at which the length of line segment PQ has a minimum value.

[Sol] The tangent line to the parabola at $P(0, 3)$ does not intersect the x-axis.
Therefore, $t \neq 0$.



From $y=\frac{1}{2}x^2+3$, $\quad y'=x$.

The equation of the tangent line at point P becomes:

$$y-\left(\frac{1}{2}t^2+3\right)=t(x-t)$$

$$\therefore y=tx-\frac{1}{2}t^2+3$$

This line intersects the x-axis at point $Q\left(\frac{t}{2}-\frac{3}{t}, 0\right)$.

Therefore, $PQ^2=\left[t-\left(\frac{t}{2}-\frac{3}{t}\right)\right]^2+\left(\frac{1}{2}t^2+3\right)^2=\frac{t^4}{4}+\frac{13}{4}t^2+12+\frac{9}{t^2}$ $\quad \cdots ①$

Letting $PQ^2=f(t)$,

$$f'(t)=t^3+\frac{13}{2}t-\frac{18}{t^3}=\frac{2t^6+13t^4-36}{2t^3}=\frac{(t^2+2)(t^2+6)(2t^2-3)}{2t^3}$$

$f'(t)=0$ when $t=\pm\frac{\sqrt{6}}{2}$

t	$\cdots$	$-\frac{\sqrt{6}}{2}$	$\cdots$	0	$\cdots$	$\frac{\sqrt{6}}{2}$	$\cdots$
$f'(t)$	$-$	0	$+$		$-$	0	$+$
$f(t)$	$\searrow$	min	$\nearrow$		$\searrow$	min.	$\nearrow$

From ①, $f\left(-\frac{\sqrt{6}}{2}\right)=f\left(\frac{\sqrt{6}}{2}\right)$

Therefore, at $t=\pm\frac{\sqrt{6}}{2}$,

PQ reaches both a relative minimum value and a minimum value.

Thus, the value of t at which the length of line segment PQ has a minimum value is:

$$t=\pm\frac{\sqrt{6}}{2}$$

KUMON

O 51 a Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

Ex. Given that $-1 \leq x \leq 0$, show that the equation $2x^5 + x^3 + 2 = 0$ has only one real number solution.

[Sol] Letting $f(x) = 2x^5 + x^3 + 2$,

$$f'(x) = 10x^4 + 3x^2 = x^2(10x^2 + 3) > 0$$

When $-\infty < x < \infty$, the graph of $f(x)$ is monotone increasing.

$\lim\limits_{x \to -\infty} f(x) = \boxed{-\infty}$ and $\lim\limits_{x \to +\infty} f(x) = \boxed{+\infty}$

Therefore, there is only one real number solution to $f(x) = 0$.

$f(-1) = \boxed{-1}$, $\quad f(0) = \boxed{2}$

Since $f(-1) < \boxed{0}$ and $f(0) > \boxed{0}$,

The given equation has only one real number solution in this domain.

Answers $-\infty$, $+\infty$, -1, 2, 0, 0

1. Given that $-1 \leq x \leq 0$, show that the equation $e^x + x = 0$ has only one real number solution.

[Sol] Letting $f(x) = e^x + x$,

$$f'(x) = e^x + 1 > 0$$

When $-\infty < x < \infty$, the graph of $f(x)$ is monotone increasing.

$\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$

Therefore, there is only one real number solution to $f(x) = 0$.

$f(-1) = e^{-1} - 1 = \dfrac{1}{e} - 1$, $\quad f(0) = 1$

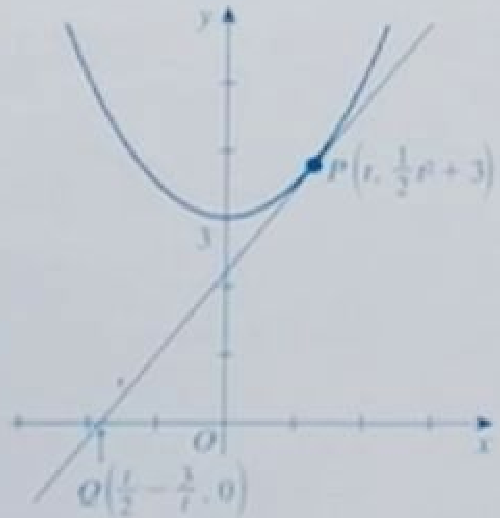
Since $f(-1) < 0$ and $f(0) > 0$,

The given equation has only one real number solution in this domain.

O 50 b

2. Given the parabola $y=\frac{1}{2}x^2+3$, letting $P\left(t,\ \frac{1}{2}t^2+3\right)$ be a point on the curve, extend the line tangent to the curve at P down to the x-axis, and denote the point of intersection with the x-axis as Q. Then, obtain the value of t at which the length of line segment PQ has a minimum value.

[Sol] The tangent line to the parabola at $P(0, 3)$ does not intersect the x-axis.
Therefore, $t \neq 0$.



From $y=\frac{1}{2}x^2+3$, $y'=x$.

The equation of the tangent line at point P becomes:

$$y-\left(\frac{1}{2}t^2+3\right)=t(x-t)$$

$$\therefore y=tx-\frac{1}{2}t^2+3$$

This line intersects the x-axis at point $Q\left(\frac{t}{2}-\frac{3}{t},0\right)$.

Therefore, $PQ^2=\left[t-\left(\frac{t}{2}-\frac{3}{t}\right)\right]^2+\left(\frac{1}{2}t^2+3\right)^2=\frac{t^4}{4}+\frac{13}{4}t^2+12+\frac{9}{t^2}$ …①

Letting $PQ^2=f(t)$,

$$f'(t)=t^3+\frac{13}{2}t-\frac{18}{t^3}=\frac{2t^6+13t^4-36}{2t^3}=\frac{(t^2+2)(t^2+6)(2t^2-3)}{2t^3}$$

$f'(t)=0$ when $t=\pm\frac{\sqrt{6}}{2}$

t	…	$-\frac{\sqrt{6}}{2}$	…	0	…	$\frac{\sqrt{6}}{2}$	…
$f'(t)$	−	0	+		−	0	+
$f(t)$	↘	min	↗		↘	min.	↗

From ①, $f\left(-\frac{\sqrt{6}}{2}\right)=f\left(\frac{\sqrt{6}}{2}\right)$.

Therefore, at $t=\pm\frac{\sqrt{6}}{2}$

PQ reaches both a relative minimum value and a minimum value.

Thus, the value of t at which the length of line segment PQ has a minimum value is:

$$t=\pm\frac{\sqrt{6}}{2}$$

KUMON

O 51 a Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0				

Ex.

Given that $-1 \leq x \leq 0$, show that the equation $2x^5 + x^3 + 2 = 0$ has only one real number solution.

[Sol] Letting $f(x) = 2x^5 + x^3 + 2$,

$$f'(x) = 10x^4 + 3x^2 = x^2(10x^2 + 3) > 0$$

When $-\infty < x < \infty$, the graph of $f(x)$ is monotone increasing.

$\lim\limits_{x \to -\infty} f(x) = \boxed{-\infty}$ and $\lim\limits_{x \to +\infty} f(x) = \boxed{+\infty}$

Therefore, there is only one real number solution to $f(x) = 0$.

$f(-1) = \boxed{-1}$, $f(0) = \boxed{2}$

Since $f(-1) < \boxed{0}$ and $f(0) > \boxed{0}$,

The given equation has only one real number solution in this domain.

Answer: $-\infty$, $+\infty$, -1, 2, 0, 0

1. Given that $-1 \leq x \leq 0$, show that the equation $e^x + x = 0$ has only one real number solution.

[Sol] Letting $f(x) = e^x + x$,

$$f'(x) = e^x + 1 > 0$$

When $-\infty < x < \infty$, the graph of $f(x)$ is monotone increasing.

$\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$

Therefore, there is only one real number solution to $f(x) = 0$.

$f(-1) = e^{-1} - 1 = \dfrac{1}{e} - 1$, $f(0) = 1$

Since $f(-1) < 0$ and $f(0) > 0$,

The given equation has only one real number solution in this domain.

O 51 b

2. Given that $-\pi \leq x \leq \pi$, show that the equation $3x + \cos 2x = 0$ has only one real number solution.

[Sol] Letting $f(x) = 3x + \cos 2x$,

$$f'(x) = 3 - 2\sin 2x > 0$$

When $-\infty < x < \infty$, the graph $f(x)$ is monotone increasing.

$\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$

Therefore, there is only one real number solution to $f(x) = 0$.

$f(-\pi) = -3\pi + 1$, $f(\pi) = 3\pi + 1$

Since $f(-\pi) < 0$ and $f(\pi) > 0$,

The given equation has only one real number solution in this domain.

3. Given that $0 < a < 1$, show that the equation $x - a\sin x = 1$ has only one real number solution.

[Sol] Letting $f(x) = x - a\sin x - 1$,

$$f'(x) = 1 - a\cos x > 0 \qquad (\because 0 < a < 1,\ |\cos x| \leq 1)$$

When $-\infty < x < \infty$, the graph of $f(x)$ is monotone increasing.

$\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$

Therefore, there is only one real number solution to $f(x) = 0$.

KUMON

O 52 a

Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

1. Given that $-2 \leq x \leq 2$, show that the equation $\dfrac{x^2-3}{x-1}=0$ has two real number solutions.

[Sol] Letting $f(x)=\dfrac{x^2-3}{x-1}$,

$$f'(x)=\frac{x^2-2x+3}{(x-1)^2}=\frac{(x-1)^2+2}{(x-1)^2}>0$$

The function is discontinuous at $x=1$.

The function is monotone increasing over each of the following intervals: $(-\infty, 1)$ and $(1, \infty)$

$\lim\limits_{x\to-\infty} f(x)=-\infty$ and $\lim\limits_{x\to+\infty} f(x)=+\infty$

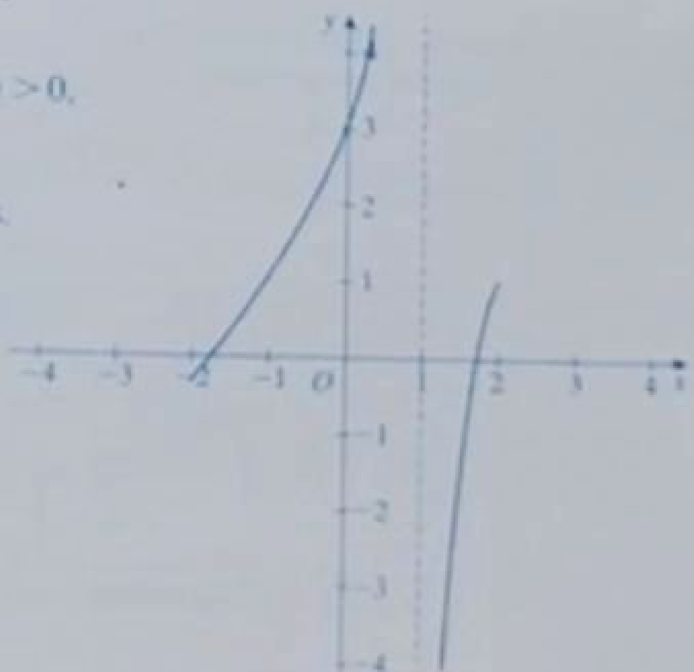
Also, $\lim\limits_{x\to1^-} f(x)=+\infty>0$ and $\lim\limits_{x\to1^+} f(x)=-\infty<0$

Therefore, there are two real number solutions to $f(x)=0$.

$f(-2)=-\dfrac{1}{3}$, $\quad f(2)=1$

Since $f(-2)<0$ and $f(2)>0$,

The given equation has two real number solutions.



O 52 b

2. Given that $-\pi \leq x \leq \pi$, show that the equation $x^3 + \tan x = 0$ has three real number solutions.

[Sol] Letting $f(x) = x^3 + \tan x$,

$$f'(x) = 3x^2 + \frac{1}{\cos^2 x} > 0$$

The function is discontinuous at $x = \pm\frac{\pi}{2}$.

The function is monotone increasing over each of the following intervals: $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\lim\limits_{x \to -\frac{3\pi}{2}} f(x) = -\infty$ and $\lim\limits_{x \to \frac{3\pi}{2}} f(x) = +\infty$

Also, $\lim\limits_{x \to -\frac{\pi}{2}-} f(x) = +\infty > 0$, $\quad \lim\limits_{x \to -\frac{\pi}{2}+} f(x) = -\infty < 0$,

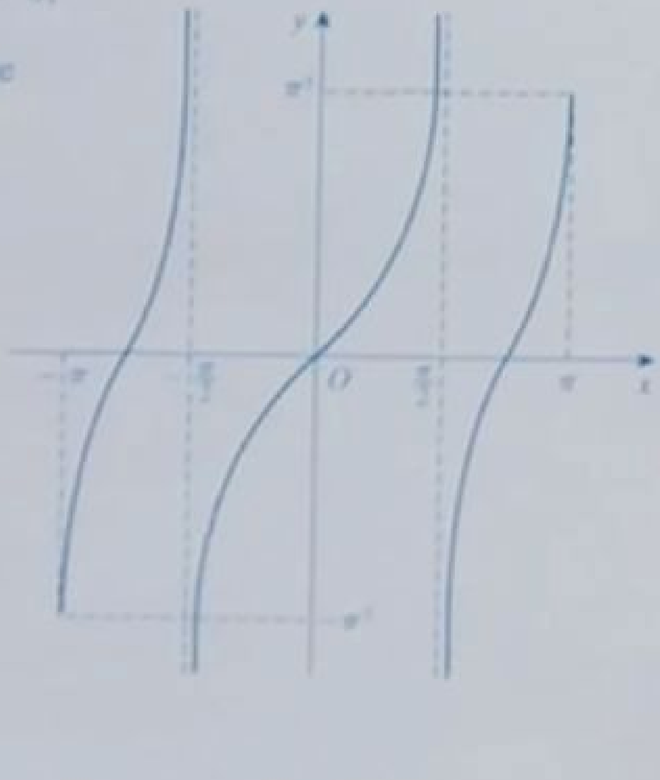
$$\lim_{x \to \frac{\pi}{2}-} f(x) = +\infty > 0 \text{ and } \lim_{x \to \frac{\pi}{2}+} f(x) = -\infty < 0$$

Therefore, there are three real number solutions to $f(x) = 0$.

$f(-\pi) = -\pi^3 < 0, \quad f(\pi) = \pi^3 > 0$

Since $f(-\pi) < 0$ and $f(\pi) > 0$,

The given equation has three real number solutions.



Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

In each exercise, determine how many real number solutions the given equation has.

(1) $x^4+4x^3-8x^2+2=0$

[Sol] Letting $f(x)=x^4+4x^3-8x^2+2$,

$$f'(x)=4x^3+12x^2-16x=4x(x^2+3x-4)$$
$$=4x(x-1)(x+4)$$

Setting up the variation table,

x	$-\infty$	$\cdots$	-4	$\cdots$	0	$\cdots$	1	$\cdots$	$+\infty$
$f'(x)$	$-$	$-$	0	$+$	0	$-$	0	$+$	$+$
$f(x)$	$+\infty$	$\searrow$	-126	$\nearrow$	2	$\searrow$	-1	$\nearrow$	$+\infty$

From the table,

The given equation has four real number solutions.

(2) $x^6-3x^2+1=0$

[Sol] Letting $f(x)=x^6-3x^2+1$,

$$f'(x)=6x^5-6x=6x(x^4-1)=6x(x^2+1)(x-1)(x+1)$$

Setting up the variation table,

x	$-\infty$	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$	$+\infty$
$f'(x)$	$-$	$-$	0	$+$	0	$-$	0	$+$	$+$
$f(x)$	$+\infty$	$\searrow$	-1	$\nearrow$	1	$\searrow$	-1	$\nearrow$	$+\infty$

From the table,

The given equation has four real number solutions.

O 53 b

(3) $6\ln x = x^2 \qquad (0 < x < e)$

[Sol] Letting $f(x) = 6\ln x - x^2$,

$$f'(x) = \frac{2(3-x^2)}{x}$$

Setting up the variation table.

x	0	$\cdots$	$\sqrt{3}$	$\cdots$	e
$f'(x)$		$+$	0	$-$	$-$
$f(x)$		$-\infty$ $\nearrow$	max.	$\searrow$	$6-e^2$

$f(\sqrt{3}) = 3(\ln 3 - 1) > 0$

$f(e) = 6 - e^2 < 0$

Thus, the given equation has two real number solutions.

(4) $x = \cos x \qquad (-\pi < x < \pi)$

[Sol] Letting $f(x) = x - \cos x$,

$f'(x) = 1 + \sin x$

Setting up the variation table.

x	$-\pi$	$\cdots$	$-\frac{\pi}{2}$	$\cdots$	π
$f'(x)$	$+$	$+$	0	$+$	$+$
$f(x)$	$-\pi+1$	$\nearrow$	$-\frac{\pi}{2}$	$\nearrow$	$\pi+1$

From the table,

The given equation has one real number solution.

Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

1. Classify the number of solutions which the equation $x^3 + ax + a = 0$ has, depending on the value of a.

[Sol] Rearranging the original equation,

$$x^3 + a(x+1) = 0 \quad \cdots ①$$

Since $x = -1$ is not a solution, dividing both sides of ① by $\boxed{x+1}$,

$$\boxed{\frac{x^3}{x+1}} + a = 0$$

Letting $f(x) = \boxed{\frac{x^3}{x+1}} + a$,

$$f'(x) = \frac{3x^2(x+1) - x^3}{(x+1)^2} = \frac{x^2(2x+3)}{(x+1)^2}$$

Setting up the variation table,

x	$-\infty$	$\cdots$	$-\frac{3}{2}$	$\cdots$	-1	$\cdots$	0	$\cdots$	$+\infty$
$f'(x)$	$-$	$-$	0	$+$	/	$+$	0	$+$	$+$
$f(x)$	$+\infty$	↘	$\frac{27}{4} + a$	↗ $+\infty$	/	$-\infty$ ↗	a	↗	$+\infty$

From the table, $f(x) = 0$ as follows:

(A) In the interval $(-1, \infty)$ at one value of x.

(B) In the interval $(-\infty, -1)$:

(a) When $\frac{27}{4} + a < 0$, in the interval $\left(-\infty, -\frac{3}{2}\right)$ at one value of x, and in the interval $\left(-\frac{3}{2}, -1\right)$ at one value of x.

(b) When $\frac{27}{4} + a = 0$, $x = -\frac{3}{2}$ Therefore, at one value of x.

(c) When $\frac{27}{4} + a > 0$, $f(x) > 0$ for all values of x in this interval. Therefore, at no value of x.

From (A) and (B), the given equation has the following number of solutions:

$$\begin{cases} \text{When } a < -\frac{27}{4}, & \text{3 solutions} \\ \text{When } a = -\frac{27}{4}, & \text{2 solutions} \\ \text{When } a > -\frac{27}{4}, & \text{1 solution} \end{cases}$$

O 54 b

2. Classify the number of solutions which the equation $x^3 - 3ax + 1 = 0$ has, depending on the value of a.

[Sol] Since $x = 0$ is not a solution, dividing both sides of the original equation by x,

$$x^2 - 3a + \frac{1}{x} = 0$$

Letting $f(x) = x^2 - 3a + \dfrac{1}{x}$,

$$f'(x) = \frac{2x^3 - 1}{x^2} = \frac{2\left(x - \sqrt[3]{\frac{1}{2}}\right)\left(x^2 + \sqrt[3]{\frac{1}{2}}x + \frac{\sqrt[3]{2}}{2}\right)}{x^2}$$

Setting up the variation table,

x	$-\infty$	$\cdots$	0	$\cdots$	$\sqrt[3]{\frac{1}{2}}$	$\cdots$	$+\infty$
$f'(x)$	$-$	$-$		$-$	0	$+$	$+$
$f(x)$	$+\infty$	$\searrow -\infty$		$+\infty \searrow$	$\frac{\sqrt[3]{2}}{2} + \sqrt[3]{2} - 3a$	$\nearrow$	$+\infty$

From the table, $f(x) = 0$ as follows:

(A) In the interval $(-\infty, 0)$ at one value of x.

(B) In the interval $(0, +\infty)$:

(a) When $\dfrac{\sqrt[3]{2}}{2} + \sqrt[3]{2} - 3a < 0$, in the interval $\left(0, \sqrt[3]{\frac{1}{2}}\right)$ at one value of x,

and in the interval $\left(\sqrt[3]{\frac{1}{2}}, +\infty\right)$ at one value of x.

(b) When $\dfrac{\sqrt[3]{2}}{2} + \sqrt[3]{2} - 3a = 0$, $x = \sqrt[3]{\frac{1}{2}}$. Therefore, at one value of x.

(c) When $\dfrac{\sqrt[3]{2}}{2} + \sqrt[3]{2} - 3a > 0$, $f(x) > 0$ for all values of x in this interval.

From (A) and (B), the given equation has the following number of solutions:

$$\begin{cases} \text{When } a > \dfrac{\sqrt[3]{2}}{2}, & \text{3 solutions} \\ \text{When } a = \dfrac{\sqrt[3]{2}}{2}, & \text{2 solutions} \\ \text{When } a < \dfrac{\sqrt[3]{2}}{2}, & \text{1 solution} \end{cases}$$

KUMON

O 55 a Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

1. Classify the number of solutions which the equation $x^{\frac{3}{2}} - mx + 1 = 0$ has, depending on the value of m.

[Sol] Rearranging the original equation,

$$x^{\frac{3}{2}} + 1 - mx = 0 \qquad \cdots ①$$

Since $x = 0$ is not a solution, dividing both sides of ① by x,

$$\sqrt{x} + \frac{1}{x} - m = 0$$

Letting $f(x) = \sqrt{x} + \frac{1}{x} - m$,

$$f'(x) = \frac{x^{\frac{3}{2}} - 2}{2x^2} \quad \text{and} \quad x > 0$$

Setting up the variation table,

x	0	$\cdots$	$2^{\frac{2}{3}}$	$\cdots$	$+\infty$
$f'(x)$		$-$	0	$+$	$+$
$f(x)$		$+\infty$ ↘	$\frac{3\sqrt[3]{2}}{2} - m$	↗	$+\infty$

Therefore, the given equation has the following number of solutions:

$$\begin{cases} \text{When } m > \dfrac{3\sqrt[3]{2}}{2}, & 2 \text{ solutions} \\ \text{When } m = \dfrac{3\sqrt[3]{2}}{2}, & 1 \text{ solution} \\ \text{When } m < \dfrac{3\sqrt[3]{2}}{2}, & 0 \text{ solutions} \end{cases}$$

O 55 b

2. Classify the number of solutions which the equation $e^x = mx$ has, depending on the value of m.

[Sol] Rearranging the original equation,

$$e^x - mx = 0$$

Letting $f(x) = e^x - mx$,

$$f'(x) = e^x - m$$

(A) When $m = 0$, $f(x) > 0$. Therefore, there is no solution.

(B) When $m < 0$, $f'(x) > 0$. Therefore, $f(x)$ is monotone increasing.
Since $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$,
$f(x)$ has one real number solution.

(C) When $m > 0$, if $x = \ln m$, then $f'(x) = 0$.

Setting up the variation table,

x	$\cdots$	$\ln m$	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	min.	$\nearrow$

Determining the relative minimum value,

$f(\ln m) = e^{\ln m} - m\ln m = m - m\ln m = m(1 - \ln m)$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (e^x - mx) = \lim_{x \to +\infty} e^x\left(1 - \frac{mx}{e^x}\right) = +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (e^x - mx) = +\infty$$

Note: $\lim_{x \to +\infty} \frac{x}{e^x} = 0$

(a) When $0 < m < e$, $f(\ln m) > 0$. Therefore, there is no real number solution.

(b) When $m = e$, $f(\ln m) = 0$. Therefore, there is one (repeated) real number solution.

(c) When $m > e$, $f(\ln m) < 0$. Therefore, there are two different real number solutions.

From (A), (B) and (C), the given equation has the following number of solutions:

$$\begin{cases} \text{When } m < 0, & 1 \text{ solution} \\ \text{When } 0 \le m < e, & 0 \text{ solutions} \\ \text{When } m = e, & 1 \text{ (repeated) solution} \\ \text{When } m > e, & 2 \text{ solutions} \end{cases}$$

Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Prove the following inequalities and state when the equality holds true.

Ex. $x^2-2x \geqq -1$

[Sol] Letting $f(x)=x^2-2x+1$,

we can prove that $f(x) \geqq 0$ by analyzing the curve of $f(x)$.

$f'(x)=\boxed{2x-2}$

x	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$

At $x=\boxed{1}$, there is a minimum value of 0.

Therefore, $f(x) \geqq \boxed{0}$ $\quad\therefore\ x^2-2x \geqq \boxed{-1}$

When $x=\boxed{1}$, the equality holds true.

Answers: $2x-2$, 1, 0, -1, 1

(1) $e^x \geqq 1+x$

[Sol] Letting $f(x)=e^x-(1+x)$,

we can prove that $f(x) \geqq 0$ by analyzing the curve of $f(x)$.

$f'(x)=e^x-1$

x	$\cdots$	0	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$

At $x=0$, there is a minimum value of 0.

Therefore, $f(x) \geqq 0$ $\quad\therefore\ e^x \geqq 1+x$

When $x=0$, the equality holds true.

O 56 b

(2) $xe^x - e^x \geq -1$

[Sol] Letting $f(x) = xe^x - e^x + 1$,
we can prove that $f(x) \geq 0$ by analyzing the curve of $f(x)$.

$$f'(x) = xe^x$$

x	$\cdots$	0	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$

At $x = 0$, there is a minimum value of 0.

Therefore, $f(x) \geq 0$ $\quad \therefore\ xe^x - e^x \geq -1$

When $x = 0$, the equality holds true.

(3) $x\ln x - x + 1 \geq 0$

[Sol] Letting $f(x) = x\ln x - x + 1$,
we can prove that $f(x) \geq 0$ by analyzing the curve of $f(x)$.

$$f'(x) = \ln x$$

x	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$

At $x = 1$, there is a minimum value of 0.

Therefore, $f(x) \geq 0$ $\quad \therefore\ x\ln x - x + 1 \geq 0$

When $x = 1$, the equality holds true.

Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Prove the following inequalities and, when applicable, state when the equality holds true.

(1) $xe^{-x} \geqq 0 \qquad (0 \leqq x \leqq 1)$

[Sol] Letting $f(x) = xe^{-x}$,

$f'(x) = e^{-x}(1-x)$

x	0	…	1
$f'(x)$	+	+	0
$f(x)$	0	↗	$\frac{1}{e}$

From the table, $f(0) = 0$.

Therefore, when $0 \leqq x \leqq 1$, $f(x) \geqq 0 \qquad \therefore xe^{-x} \geqq 0$

When $x = 0$, the equality holds true.

(2) $\frac{x^2}{2} + \frac{1}{x} > 1 \qquad (x > 0)$

[Sol] Letting $f(x) = \frac{x^2}{2} + \frac{1}{x} - 1$,

$$f'(x) = x - \frac{1}{x^2} = \frac{(x-1)(x^2+x+1)}{x^2}$$

x	0	…	1	…
$f'(x)$		−	0	+
$f(x)$		↘	$\frac{1}{2}$	↗

From the table, $f(1) = \frac{1}{2}$

Therefore, when $x > 0$, $f(x) > 0 \qquad \therefore \frac{x^2}{2} + \frac{1}{x} > 1$

O 57 b

(3) $\sin x > \frac{2}{\pi}x \qquad \left(0 < x < \frac{\pi}{2}\right)$

[Sol] Letting $f(x) = \sin x - \frac{2}{\pi}x$,

$$f'(x) = \cos x - \frac{2}{\pi}$$

Letting α be the solution of $f'(x)$,

x	0	$\cdots$	α	$\cdots$	$\frac{\pi}{2}$
$f'(x)$	$+$	$+$	0	$-$	$-$
$f(x)$	0	↗	max.	↘	0

From the table, $f(0) = f\left(\frac{\pi}{2}\right) = 0$.

Therefore, when $0 < x < \frac{\pi}{2}$, $f(x) > 0 \qquad \therefore \sin x > \frac{2}{\pi}x$

(4) $\sin x \geq x\cos x \qquad (0 \leq x \leq \pi)$

[Sol] Letting $f(x) = \sin x - x\cos x$,

$$f'(x) = \cos x - \cos x + x\sin x = x\sin x \geq 0$$

x	0	$\cdots$	π
$f'(x)$	0	$+$	0
$f(x)$	0	↗	π

From the table, $f(0) = 0$.

Therefore, when $0 \leq x \leq \pi$, $f(x) \geq 0 \qquad \therefore \sin x \geq x\cos x$

When $x = 0$, the equality holds true.

KUMON

O 58 a Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69% -
(mistakes) 0	-	-	1	2-

Prove the following inequalities.

(1) $e^x > 1 + x + \dfrac{x^2}{2}$ $(x > 0)$

[Sol] Letting $f(x) = e^x - \left(1 + x + \dfrac{x^2}{2}\right)$.

If we show that $f''(x) > 0$, we can conclude that $f'(x)$ is monotone increasing.

If $f(0) = 0$, and $f'(x) > 0$ we can conclude that $f(x) > 0$.

$$f'(x) = e^x - 1 - x$$

$$f''(x) = e^x - 1 > 0 \quad (\because x > 0)$$

Therefore, when $x > 0$, $f'(x)$ is monotone increasing.

Since $f'(0) = 0$, $f'(x) > 0$.

Thus, when $x > 0$, $f(x)$ is monotone increasing.

Since $f(0) = 0$, $f(x) > 0$.

$$\therefore e^x > 1 + x + \frac{x^2}{2}$$

(2) $x \sin x < 2(1 - \cos x)$ $\left(0 < x \leq \dfrac{\pi}{2}\right)$

[Sol] Letting $f(x) = 2(1 - \cos x) - x \sin x$.

$$f'(x) = \sin x - x \cos x$$

$$f''(x) = x \sin x > 0$$

Therefore, when $0 < x \leq \dfrac{\pi}{2}$, $f'(x)$ is monotone increasing.

Since $f'(0) = 0$, and since when $0 \leq x \leq \dfrac{\pi}{2}$, $f'(x)$ is continuous: $f'(x) \geq 0$

Thus, when $0 \leq x \leq \dfrac{\pi}{2}$, $f(x)$ is monotone increasing.

Since $f(0) = 0$, when $0 < x \leq \dfrac{\pi}{2}$, $f(x) > 0$.

$$\therefore x \sin x < 2(1 - \cos x)$$

O 58 b

(3) $1-\frac{x}{2}+\frac{3x^2}{8}>\frac{1}{\sqrt{1+x}}$ $(x>0)$

[Sol] Letting $f(x)=1-\frac{x}{2}+\frac{3x^2}{8}-\frac{1}{\sqrt{1+x}}$,

$$f'(x)=-\frac{1}{2}+\frac{3x}{4}+\frac{1}{2\sqrt{(1+x)^3}}$$

$$f''(x)=\frac{3}{4}\left[1-\frac{1}{\sqrt{(1+x)^5}}\right]$$

Since $x>0$ and $f''(x)>0$, $f'(x)$ is monotone increasing.
Since $f'(0)=0$, $f'(x)>0$. Therefore, $f(x)$ is monotone increasing.
Since $f(0)=0$, when $x>0$, $f(x)>0$.

$$\therefore 1-\frac{x}{2}+\frac{3x^2}{8}>\frac{1}{\sqrt{1+x}}$$

(4) $x-\frac{x^3}{6}<\sin x<x$ $(x>0)$ (**Hint:** First show that $\sin x<x$.)

[Sol] Letting $f(x)=x-\sin x$, and letting $g(x)=\sin x-\left(x-\frac{x^3}{6}\right)$,

$$f'(x)=1-\cos x\geq 0$$

Therefore, $f(x)$ is monotone increasing.
Since $f(0)=0$, when $x>0$, $f(x)>0$. $\therefore \sin x<x$ …①

$$g'(x)=\cos x-1+\frac{x^2}{2}$$

$$g''(x)=-\sin x+x=f(x)>0 \quad (\because x>0)$$

Therefore, when $x>0$, $g'(x)$ is monotone increasing.
Since $g'(0)=0$, when $x>0$, $g'(x)>0$.

Thus, when $x>0$, $g(x)$ is monotone increasing.

Since $g(0)=0$, when $x>0$, $g(x)>0$. $\therefore x-\frac{x^3}{6}<\sin x$ …②

From ① and ②, $x-\frac{x^3}{6}<\sin x<x$

KUMON

O 59 a Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	-	1-

Prove the following inequalities.

(1) $\ln(1+x) > \dfrac{x}{1+x}$ $(x>0)$

[Sol] Letting $f(x) = \ln(1+x) - \dfrac{x}{1+x}$,

$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}$$

When $x>0$, $f'(x)>0$. Therefore, $f(x)$ is monotone increasing.
Since $f(0)=0$, when $x>0$, $f(x)>0$.

$$\therefore \ln(1+x) > \frac{x}{1+x}$$

(2) $2x - x^2 < \ln(1+x)^2 < 2x$ $(x>0)$

[Sol] Letting $f(x) = \ln(1+x)^2 - (2x - x^2)$,

$$f'(x) = \frac{2}{1+x} - (2-2x) = \frac{2x^2}{1+x}$$

When $x>0$, $f'(x)>0$. Therefore, $f(x)$ is monotone increasing.
Since $f(0)=0$, when $x>0$, $f(x)>0$

$\therefore 2x - x^2 < \ln(1+x)^2$ …①

Letting $g(x) = 2x - \ln(1+x)^2$,

$$g'(x) = 2 - \frac{2}{1+x} = \frac{2x}{1+x}$$

When $x>0$, $g'(x)>0$. Therefore, $g(x)$ is monotone increasing.
Since $g(0)=0$, when $x>0$, $g(x)>0$

$\therefore \ln(1+x)^2 < 2x$ …②

From ① and ②,

$2x - x^2 < \ln(1+x)^2 < 2x$

O 59 b

(3) $x-1>\sqrt{x}\ln x$ $\qquad (x>1)$

[Sol] Letting $f(x)=x-1-\sqrt{x}\ln x$,

$$f'(x)=\frac{2\sqrt{x}-\ln x-2}{2\sqrt{x}}$$

$$f''(x)=\frac{\ln x}{4x\sqrt{x}}>0 \qquad (\because x>1)$$

Since $x>1$ and $f''(x)>0$, $f'(x)$ is monotone increasing.
Since $f'(1)=0$, $f'(x)>0$. Therefore, $f(x)$ is monotone increasing.
Since $f(1)=0$, when $x>1$, $f(x)>0$.
$\therefore x-1>\sqrt{x}\ln x$

KUMON

O 60 a Applications of Differential Calculus 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	[illegible]

1. Classify the number of solutions which the equation $\ln x = mx$ has, depending on the value of m.

[Sol] Rearranging the original equation,

$$\ln x - mx = 0 \qquad \cdots ①$$

Since $x = 0$ is not a solution, dividing both sides of ① by x,

$$\frac{\ln x}{x} - m = 0$$

Letting $f(x) = \dfrac{\ln x}{x} - m$,

$$f'(x) = \frac{(\ln x)'x - (\ln x)x'}{x^2} = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}, \text{ and } x > 0$$

x	0	$\cdots$	e	$\cdots$	$+\infty$
$f'(x)$		$+$	0	$-$	$-$
$f(x)$		$-\infty$ ↗	$\frac{1}{e} - m$	↘	$-m$

From the table,

$$\begin{cases} \text{When } 0 < m < \frac{1}{e}, & \text{2 solutions} \\ \text{When } m \leqq 0,\ m = \frac{1}{e}, & \text{1 solution} \\ \text{When } m > \frac{1}{e}, & \text{0 solutions} \end{cases}$$

O 60 b

2. Prove the following inequality.

$$x\ln x + 1 > 2x - x^2 \qquad (x > 1)$$

[Sol] Letting $f(x) = x\ln x + x^2 - 2x + 1$,

$$\begin{aligned} f'(x) &= (\ln x + 1) + 2x - 2 \\ &= \ln x + 2x - 1 \end{aligned}$$

$$f''(x) = \frac{1}{x} + 2 > 0$$

Since $x > 1$, and $f''(x) > 0$, $f'(x)$ is monotone increasing.
Since $f'(1) = 1$, $f'(x) > 0$. Therefore, $f(x)$ is monotone increasing.
Since $f(1) = 0$, when $x > 1$, $f(x) > 0$.

$\therefore\ x\ln x + 1 > 2x - x^2$

Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

Rolle's Theorem

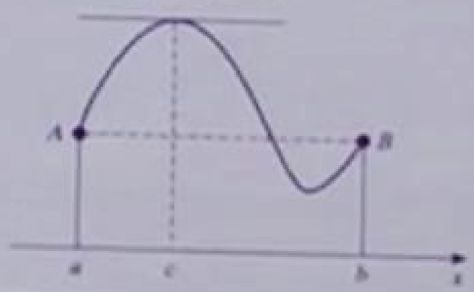
Given that a function $f(x)$ satisfies the following three conditions,

(a) $f(x)$ is continuous over the closed interval $[a, b]$.

(b) $f(x)$ has a derivative $f'(x)$ over the open interval (a, b).

(c) $f(a) = f(b)$

Then, there is at least one value c (where $a < c < b$), at which $f'(c) = 0$.



If points A and B have the same y-coordinate, then there will be at least one point on the curve whose tangent line will have a slope of 0. (i.e. there is a c such that $f'(c) = 0$)

1. In each exercise, find the value(s) of c which satisfy the conditions of Rolle's Theorem for the given function.

(1) $f(x) = x^2 + x - 2 \qquad (-2 \leq x \leq 1)$

[Sol] $f'(x) = 2x + 1$

$f'(c) = 2c + 1 = 0$

$\therefore c = -\frac{1}{2}$

(2) $f(x) = x^3 - x \qquad (-1 \leq x \leq 1)$

[Sol] $f'(x) = 3x^2 - 1$

$f'(c) = 3c^2 - 1 = 0$

$\therefore c = \pm\frac{\sqrt{3}}{3}$

(3) $f(x) = \sin\left(2x + \frac{\pi}{3}\right) \qquad \left(0 \leq x \leq \frac{\pi}{6}\right)$

[Sol] $f'(x) = 2\cos\left(2x + \frac{\pi}{3}\right)$

$f'(c) = 2\cos\left(2c + \frac{\pi}{3}\right) = 0$

$2c + \frac{\pi}{3} = \frac{\pi}{2}$

$\therefore c = \frac{\pi}{12}$

Notation: $[a, b]$ denotes the closed interval where $a \leq x \leq b$.
(a, b) denotes the open interval where $a < x < b$.

2. Given that $f(x)$ is a fourth degree polynomial equation with real number coefficients, prove the following conditions when $f(x)=0$.

(1) If a and b are different real number solutions, then $f'(x)=0$ has at least one solution between a and b.

[Sol] Since $f(x)$ is a fourth degree polynomial, it is differentiable for all values of x.

Since a and b are different real number solutions,

$f(a)=f(b)=0$

From Rolle's Theorem,

$f'(x)=0$ has at least one solution between a and b.

(2) If $f(x)$ has four different real number solutions, then $f'(x)=0$ has three different real number solutions.

[Sol] Letting a, b, c and d be the four solutions, (where $a<b<c<d$),

$f(a)=f(b)=f(c)=f(d)=0$

From Rolle's Theorem,

$f'(x)=0$ has at least one solution between each of the following:

a and b, b and c, c and d

Since $f'(x)=0$ is a third degree polynomial, it cannot have more than three solutions.

Therefore, $f'(x)=0$ has three different real number solutions.

KUMON

O 62 a

Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%-

Mean Value Theorem

Given that a function $f(x)$ satisfies the following two conditions,

(a) $f(x)$ is continuous over the closed interval $[a, b]$.

(b) $f(x)$ has a derivative $f'(x)$ over the open interval (a, b).

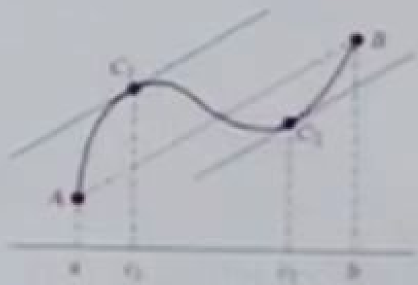
Then, there is at least one value c (where $a < c < b$), at which:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

1. Use Rolle's Theorem to prove the Mean Value Theorem.

[Sol] Letting $k = \dfrac{f(b) - f(a)}{b - a}$, …①

show that $k = f'(c)$.



Note: The slope of the tangent line at C_1 is the same as the slope of the line connecting points A and B.

Rearranging ①,

$f(b) - f(a) - k(b - a) = 0$ …②

Replacing a with x in ②, and

Letting $F(x) = f(b) - f(x) - k(b - x)$,

From ②,

$$F(a) = F(b) = \boxed{0}$$

$$F'(x) = \boxed{-f'(x) + k}$$

From Rolle's Theorem, given the above function $F(x)$, there is at least one value c (where $a < c < b$) at which $F'(c) = 0$.

Since $F'(c) = \boxed{-f'(c) + k}$,

$$f'(c) = \boxed{k}$$

Therefore, there is at least one value c (where $a < c < b$) at which:

$$\frac{f(b) - f(a)}{b - a} = \boxed{f'(c)}$$

O 62 b

2. In each exercise, find the value of c which satisfies the conditions of the Mean Value Theorem for the given function.

(1) $f(x) = 2x^2 + x - 2 \qquad (0 \le x \le 3)$

[Sol] $f'(x) = \boxed{4x+1}$

$f'(c) = \boxed{4c+1}$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(\boxed{0})}{3-\boxed{0}} = \frac{19+2}{3} = 7$$

Letting $f'(c) = \dfrac{f(b)-f(a)}{b-a}$,

$4c + 1 = 7$

$\therefore c = \dfrac{3}{2}$

(2) $f(x) = x^3 - x^2 \qquad (1 \le x \le 4)$

[Sol] $f'(x) = 3x^2 - 2x$

$f'(c) = 3c^2 - 2c$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(1)}{4-1} = \frac{48-0}{3} = 16$$

Letting $f'(c) = \dfrac{f(b)-f(a)}{b-a}$,

$3c^2 - 2c = 16$

$3c^2 - 2c - 16 = 0$

$(3c-8)(c+2) = 0$

$\therefore c = \dfrac{8}{3}$

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O 63 a Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

1. In each exercise, find the value of c which satisfies the conditions of the Mean Value Theorem for the given function.

(1) $f(x) = x^2$

[Sol] $f'(x) = 2x$

$$\frac{f(b)-f(a)}{b-a} = \frac{b^2-a^2}{b-a} = b+a$$

$f'(c) = 2c$

Therefore, $b + a = 2c$

$$\therefore c = \frac{a+b}{2}$$

(2) $f(x) = x^3$ (a and b are positive.)

[Sol] $$\frac{f(b)-f(a)}{b-a} = \frac{b^3-a^3}{b-a} = b^2+ab+a^2$$

$f'(c) = 3c^2$

Therefore, $3c^2 = b^2 + ab + a^2$

Thus, from $0 < a < c < b$,

$$c = \sqrt{\frac{a^2+ab+b^2}{3}}$$

(3) $f(x) = \dfrac{1}{x}$ (a and b have the same sign.)

[Sol] $$\frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab}$$

$$f'(c) = -\frac{1}{c^2}$$

Therefore, $-\dfrac{1}{c^2} = -\dfrac{1}{ab}$ $\quad\therefore c = \pm\sqrt{ab}$

Thus,

When $0 < a < b$, $c = \sqrt{ab}$

When $a < b < 0$, $c = -\sqrt{ab}$

O 63 b

2. Given a function $f(x)$ such that $f(1) = 3$ and $f'(x) \leq 2$ for all values of x, determine the maximum value of $f(5)$.

[Sol] Using the Mean Value Theorem over the interval [1, 5].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(b) = f(a) + f'(c) \cdot \boxed{(b - a)}$$

$$f(5) = f(1) + f'(c) \cdot \boxed{4}$$

$$= 3 + 4f'(c)$$

Since $f'(x) \leq 2$ for all values of x,
the maximum value of $f(5)$ is $3 + 4 \cdot 2 = 11$.

3. Given a function $f(x)$ such that $f(-2) = -1$ and $f'(x) \leq 7$ for all values of x, determine the maximum value of $f(-1)$.

[Sol] Using the Mean Value Theorem over the interval $[-2, -1]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(b) = f(a) + f'(c) \cdot (b - a)$$

$$f(-1) = f(-2) + f'(c) \cdot 1$$

$$= -1 + f'(c)$$

Since $f'(x) \leq 7$ for all values of x,
the maximum value of $f(-1)$ is $-1 + 7 = 6$.

Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

Use the Mean Value Theorem to prove the following inequalities.

(1) When $0<a<b$, $\quad 1-\dfrac{a}{b}<\ln\dfrac{b}{a}<\dfrac{b}{a}-1$

[Sol] Letting $f(x)=\ln x$,

$$f'(x)=\boxed{\frac{1}{x}}$$

$$f'(c)=\boxed{\frac{1}{c}}$$

Since $f(x)$ is continuous over the closed interval $[a, b]$ and $f(x)$ has a derivative $f'(x)$ over the open interval (a, b),

Using the Mean Value Theorem,

There is at least one value c (where $a<c<b$), …①

at which: $f'(c)=\boxed{\dfrac{f(b)-f(a)}{b-a}}$

Since $\dfrac{f(b)-f(a)}{b-a}=\dfrac{\ln b-\boxed{\ln a}}{b-a}$,

Letting $f'(c)=\dfrac{f(b)-f(a)}{b-a}$,

$$\frac{1}{c}=\frac{\ln b-\ln a}{b-a} \quad \text{…②}$$

From ①,

$$\frac{1}{\boxed{b}}<\frac{1}{c}<\frac{1}{\boxed{a}} \quad \text{…③}$$

Substituting ② into ③,

$$\frac{1}{b}<\frac{\ln b-\ln a}{b-a}<\frac{1}{a}$$

Since $b-a>0$,

$$\frac{b-a}{b}<\ln b-\ln a<\frac{b-a}{a}$$

$$\therefore 1-\frac{a}{b}<\ln\frac{b}{a}<\frac{b}{a}-1$$

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(2) When $0<a<b$, $\frac{1}{b^2}<\frac{1}{ab}<\frac{1}{a^2}$

[Sol] Letting $f(x)=\frac{1}{x}$,

$$f'(x)=-\frac{1}{x^2}$$

$$f'(c)=-\frac{1}{c^2}$$

Since $f(x)$ is continuous over the closed interval $[a, b]$ and $f(x)$ has a derivative $f'(x)$ over the open interval (a, b),

Using the Mean Value Theorem,
There is at least one value c (where $a<c<b$). ... ①

at which: $f'(c)=\frac{f(b)-f(a)}{b-a}$

Since $\frac{f(b)-f(a)}{b-a}=\frac{\frac{1}{b}-\frac{1}{a}}{b-a}=-\frac{1}{ab}$,

Letting $f'(c)=\frac{f(b)-f(a)}{b-a}$,

$$\frac{1}{c^2}=\frac{1}{ab} \quad \text{... ②}$$

From ①,

$$\frac{1}{b^2}<\frac{1}{c^2}<\frac{1}{a^2} \quad \text{... ③}$$

Substituting ② into ③,

$$\frac{1}{b^2}<\frac{1}{ab}<\frac{1}{a^2}$$

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O 65 a Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

Use the Mean Value Theorem to prove the following inequalities.

(1) When $a>0$, $\dfrac{1}{a+1}<\ln(a+1)-\ln a<\dfrac{1}{a}$

[Sol] Letting $f(x)=\ln x$,

$$f'(x)=\frac{1}{x}$$

$$f'(c)=\frac{1}{c}$$

Since $f(x)$ is continuous over the closed interval $[a, a+1]$ and $f(x)$ has a derivative $f'(x)$ over the open interval $(a, a+1)$,

Using the Mean Value Theorem,

There is at least one value c (where $a<c<a+1$), —①

at which: $f'(c)=\dfrac{f(a+1)-f(a)}{(a+1)-a}$

Since $\dfrac{f(a+1)-f(a)}{(a+1)-a}=\dfrac{\ln(a+1)-\ln a}{(a+1)-a}=\ln(a+1)-\ln a$,

Letting $f'(c)=\dfrac{f(a+1)-f(a)}{(a+1)-a}$,

$$\frac{1}{c}=\ln(a+1)-\ln a \quad \text{—②}$$

From ①,

$$\frac{1}{a+1}<\frac{1}{c}<\frac{1}{a} \quad \text{—③}$$

Substituting ② into ③,

$$\frac{1}{a+1}<\ln(a+1)-\ln a<\frac{1}{a}$$

O 65 b

(2) When $0 \leq q < p$, $n \geq 2$, $\quad p^n - q^n < np^{n-1}(p-q)$

[Sol] Letting $f(x) = x^n$,

$$f'(x) = nx^{n-1}$$
$$f'(c) = nc^{n-1}$$

Since $f(x)$ is continuous over the closed interval $[q, p]$ and $f(x)$ has a derivative $f'(x)$ over the open interval (q, p).

Using the Mean Value Theorem.

There is at least one value c (where $q < c < p$). $\quad$ ...①

at which: $f'(c) = \dfrac{f(p) - f(q)}{p - q}$

Since $\dfrac{f(p) - f(q)}{p - q} = \dfrac{p^n - q^n}{p - q}$,

Letting $f'(c) = \dfrac{f(p) - f(q)}{p - q}$,

$$nc^{n-1} = \frac{p^n - q^n}{p - q} \quad \text{...②}$$

From ①,

$$nc^{n-1} < np^{n-1} \quad \text{...③}$$

Substituting ② into ③,

$$\frac{p^n - q^n}{p - q} < np^{n-1}$$
$$p^n - q^n < np^{n-1}(p - q) \qquad (\because p - q > 0)$$

Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

Obtain the first order derivative of each of the following functions.

(1) $y = xe^{-3x}$

[Sol] $y' = (x)'e^{-3x} + x(e^{-3x})'$

$= e^{-3x} - 3xe^{-3x}$

$= e^{-3x}(1-3x)$

(2) $y = \cos(-2x) + \tan(3x^2)$

[Sol] $y' = -\sin(-2x)(-2x)' + \dfrac{1}{\cos^2(3x^2)}(3x^2)'$

$= 2\sin(-2x) + \dfrac{6x}{\cos^2(3x^2)}$

(3) $y = \ln\left(\sqrt{x^2-1} - x\right)$

[Sol] $y' = \dfrac{\left(\sqrt{x^2-1}-x\right)'}{\sqrt{x^2-1}-x}$

$= \dfrac{\dfrac{x}{\sqrt{x^2-1}} - 1}{\sqrt{x^2-1}-x}$

$= -\dfrac{1}{\sqrt{x^2-1}}$ $\left[= -\dfrac{\sqrt{(x-1)(x+1)}}{(x-1)(x+1)}\right]$

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(4) $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

[Sol] $y' = \dfrac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2}$

$= -\dfrac{4}{(e^x - e^{-x})^2}$

Alternate Solution

$y' = \dfrac{2e^{2x}(e^{2x} - 1) - (e^{2x} + 1)(2e^{2x})}{(e^{2x} - 1)^2}$

$= -\dfrac{4e^{2x}}{(e^{2x} - 1)^2}$

(5) $y = x^3 \log_4 x$

[Sol] $y = x^3 \dfrac{\ln x}{\ln 4}$

$y' = \dfrac{1}{\ln 4}(x^3 \ln x)'$

$= \dfrac{1}{\ln 4}\left(3x^2 \cdot \ln x + x^3 \cdot \dfrac{1}{x}\right)$

$= \dfrac{x^2(3\ln x + 1)}{\ln 4}$

(6) $y = x^{-3x}$

[Sol] Taking the natural logarithm of both sides,

$\ln y = -3x \ln x$

Differentiating both sides with respect to x,

$\dfrac{y'}{y} = -3\ln x - 3$

$\therefore y' = -3x^{-3x}(\ln x + 1)$

Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%−
(mistakes) 0	−	1	−	2−

In each exercise, obtain the value of $\frac{dy}{dx}$ of the given function.

(1) $(2x-3)^2 = y^2 + x$

[Sol] Differentiating both sides with respect to x,

$$4(2x-3) = 2y\frac{dy}{dx} + 1$$

$$\therefore \frac{dy}{dx} = \frac{8x-13}{2y}$$

(2) $\cos x - \cos y = 1$

[Sol] Differentiating both sides with respect to x,

$$-\sin x + (\sin y)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{\sin x}{\sin y}$$

(3) $\ln(x-y) = -2x$

[Sol] Differentiating both sides with respect to x,

$$\frac{1-\frac{dy}{dx}}{x-y} = -2$$

$$\therefore \frac{dy}{dx} = 2x - 2y + 1$$

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(4) $x=\dfrac{t^2+1}{t^2-1}, \qquad y=\dfrac{3t}{t^2-1}$

[Sol] $\dfrac{dx}{dt}=\dfrac{2t(t^2-1)-(t^2+1)\cdot 2t}{(t^2-1)^2}=-\dfrac{4t}{(t^2-1)^2},$

$\dfrac{dy}{dt}=\dfrac{3(t^2-1)-3t\cdot 2t}{(t^2-1)^2}=-\dfrac{3(t^2+1)}{(t^2-1)^2}$

$\therefore \dfrac{dy}{dx}=\dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\dfrac{3(t^2+1)}{4t}$

(5) $x=a(\sin t-t\cos t), \qquad y=a(\cos t-t\sin t)$

[Sol] $\dfrac{dx}{dt}=a[\cos t-(\cos t-t\sin t)]=at\sin t$

$\dfrac{dy}{dt}=a[-\sin t-(\sin t+t\cos t)]=-a(2\sin t+t\cos t)$

$\therefore \dfrac{dy}{dx}=-\dfrac{a(2\sin t+t\cos t)}{at\sin t}=-\dfrac{2}{t}-\dfrac{1}{\tan t}$

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O 68 a Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

1. For the given function, create a variation table indicating where the curve is concave up and concave down, and note the point(s) of inflection. Then, state the asymptote(s), (if any), and draw the graph.

$$y = x^2 - \frac{2}{x}$$

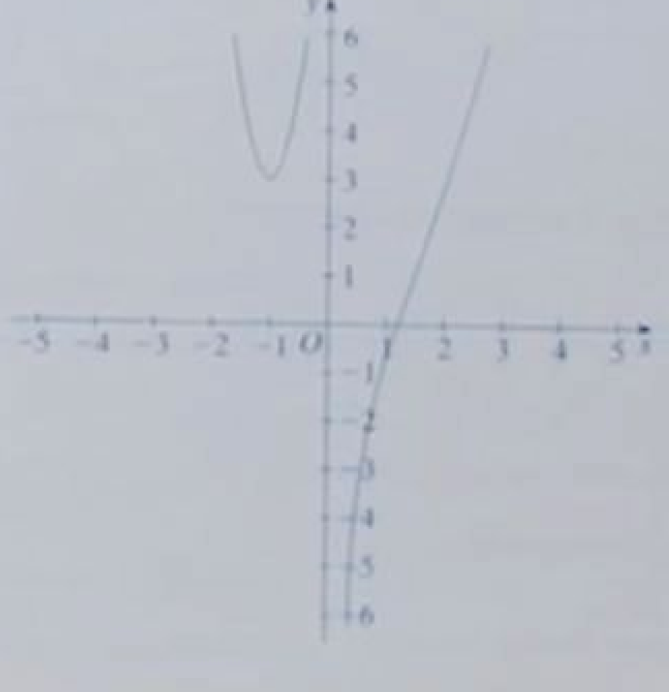
[Sol] $y' = 2x + \frac{2}{x^2} = \frac{2x^3+2}{x^2} = \frac{2(x+1)(x^2-x+1)}{x^2}$

$y'' = \frac{2x^3-4}{x^3} = \frac{2(x^3-2)}{x^3}$

x	$\cdots$	-1	$\cdots$	0	$\cdots$	$\sqrt[3]{2}$	$\cdots$
y'	$-$	0	$+$		$+$	$+$	$+$
y''	$+$	$+$	$+$		$-$	0	$+$
y	↘	3	↗		↗	i.p.	↗

Since $\lim_{x \to -0} y = -\infty$ and $\lim_{x \to +0} y = +\infty$,

The asymptote is: $x = 0$



O 68 b

2. Find the equations of the tangent line and the normal line to each of the following curves at the given point. Then, graph the curve, the tangent line and the normal line.

(1) $y=\dfrac{x}{e^x}$ $\qquad (0, 0)$

[Sol] Letting $f(x)=\dfrac{x}{e^x}$,

$$f'(x)=\frac{1-x}{e^x}$$

$$f'(0)=1$$

The equation of the tangent line is:

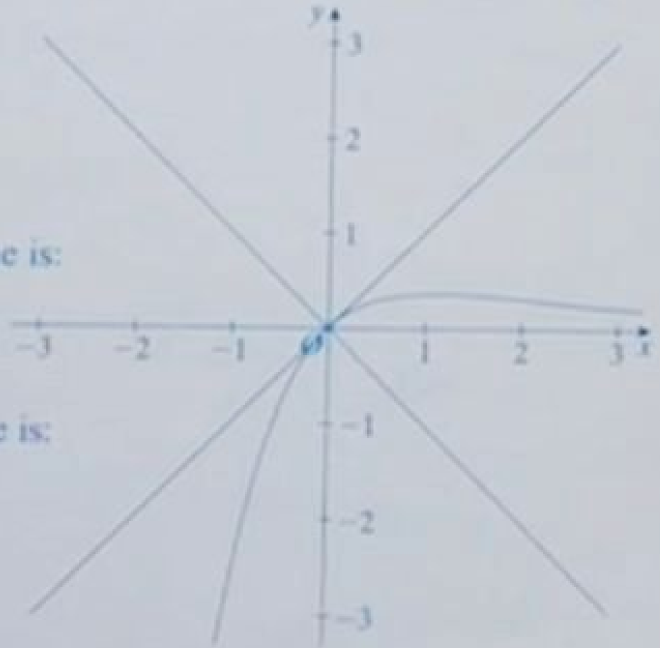
$$y-0=1(x-0)$$

$$y=x$$

The equation of the normal line is:

$$y-0=-1(x-0)$$

$$y=-x$$



(2) $y=\dfrac{1}{\sqrt{12-3x^2}}$ $\qquad \left(-1, \dfrac{1}{3}\right)$

[Sol] Letting $f(x)=\dfrac{1}{\sqrt{12-3x^2}}$,

$$f'(x)=\frac{3x}{(12-3x^2)\sqrt{12-3x^2}}$$

$$f'(-1)=-\frac{1}{9}$$

The equation of the tangent line is:

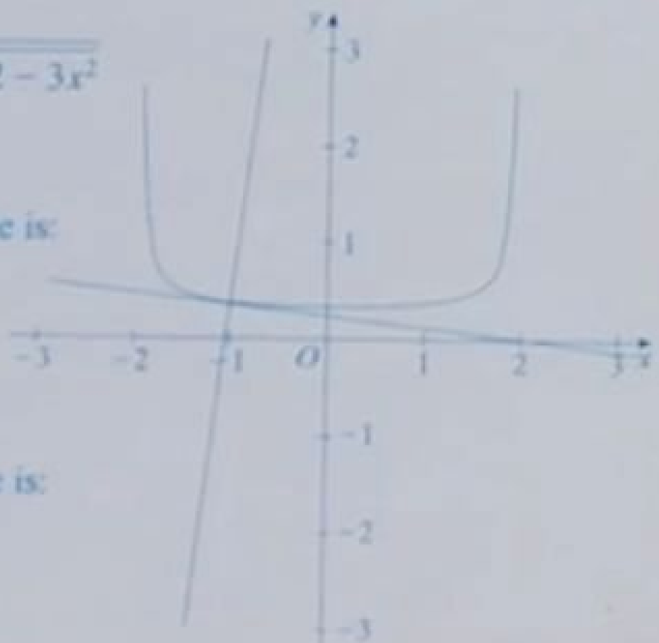
$$y-\frac{1}{3}=-\frac{1}{9}(x+1)$$

$$y=-\frac{1}{9}(x-2)$$

The equation of the normal line is:

$$y-\frac{1}{3}=9(x+1)$$

$$y=\frac{1}{3}(27x+28)$$



Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	–	–	1–

1. In each exercise, obtain the maximum and minimum values of the given function.

(1) $f(x)=\dfrac{3x}{4+x^2}$ $\quad(-4\leq x\leq 4)$

[Sol] $f'(x)=-\dfrac{3(x^2-4)}{(x^2+4)^2}$

$f'(x)=0$ when $x=\pm 2$

x	-4	$\cdots$	-2	$\cdots$	2	$\cdots$	4
$f'(x)$	$-$	$-$	0	$+$	0	$-$	$-$
$f(x)$	$-\dfrac{3}{5}$	$\searrow$	$-\dfrac{3}{4}$	$\nearrow$	$\dfrac{3}{4}$	$\searrow$	$\dfrac{3}{5}$

From the table,

The maximum value is $\dfrac{3}{4}$, (at $x=2$).

The minimum value is $-\dfrac{3}{4}$, (at $x=-2$).

(2) $f(x)=\dfrac{\ln x}{x}$ $\quad(1\leq x\leq e^3)$

[Sol] $f'(x)=\dfrac{1-\ln x}{x^2}$

$f'(x)=0$ when $x=e$

x	1	$\cdots$	e	$\cdots$	e^3
$f'(x)$	$+$	$+$	0	$-$	$-$
$f(x)$	0	$\nearrow$	$\dfrac{1}{e}$	$\searrow$	$\dfrac{3}{e^3}$

From the table,

The maximum value is $\dfrac{1}{e}$, (at $x=e$).

The minimum value is 0, (at $x=1$).

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2. Given that x and y are positive real numbers such that $x^2 - 3x + y^2 = 0$, obtain the maximum value of the product xy.

[Sol] From $x^2 - 3x + y^2 = 0$,

$y = \sqrt{3x - x^2}$, $\quad (0 < x < 3)$ $\quad \cdots ①$

From ①, since $y = \sqrt{3x - x^2}$,

$$xy = x\sqrt{3x - x^2}$$

Letting $f(x) = x\sqrt{3x - x^2}$,

$$f'(x) = \sqrt{3x - x^2} + x \cdot \frac{3 - 2x}{2\sqrt{3x - x^2}} = -\frac{x(4x - 9)}{2\sqrt{3x - x^2}}$$

x	0	$\cdots$	$\frac{9}{4}$	$\cdots$	3
$f'(x)$		$+$	0	$-$	
$f(x)$	0	$\nearrow$	$\frac{27\sqrt{3}}{16}$	$\searrow$	0

Over the domain $0 < x < 3$,

$f'(x) = 0$ when $x = \frac{9}{4}$. At this value, $y = \frac{3\sqrt{3}}{4}$.

Therefore, the maximum value of xy is $\frac{27\sqrt{3}}{16}$.

Applications of Differential Calculus 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

1. Classify the number of solutions which the equation $2x^2 - mx + 3 = 0$ has, depending on the value of m.

[Sol] Rearranging the original equation,

$$2x^2 + 3 - mx = 0 \quad \cdots ①$$

Since $x = 0$ is not a solution, dividing both sides of ① by x,

$$2x + \frac{3}{x} - m = 0$$

Letting $f(x) = 2x + \frac{3}{x} - m$,

$$f'(x) = \frac{2x^2 - 3}{x^2}$$

x	$-\infty$	$\cdots$	$-\frac{\sqrt{6}}{2}$	$\cdots$	0	$\cdots$	$\frac{\sqrt{6}}{2}$	$\cdots$	$+\infty$
$f'(x)$	$+$	$+$	0	$-$	/	$-$	0	$+$	$+$
$f(x)$	$-\infty$	↗	$-2\sqrt{6} - m$	↘	/	↘	$2\sqrt{6} - m$	↗	$+\infty$

Therefore, the given equation has the following number of solutions:

$$\begin{cases} \text{When } m < -2\sqrt{6},\ m > 2\sqrt{6}, & \text{2 solutions} \\ \text{When } m = \pm 2\sqrt{6}, & \text{1 solution} \\ \text{When } -2\sqrt{6} < m < 2\sqrt{6}, & \text{0 solutions} \end{cases}$$

O 70 b

2. Given the function $f(x)=\frac{1}{x^2}$, find the value of c which satisfies the conditions of the Mean Value Theorem for the function.
(a and b are positive.)

[Sol] $\frac{f(b)-f(a)}{b-a}=\frac{\frac{1}{b^2}-\frac{1}{a^2}}{b-a}=-\frac{a+b}{a^2b^2}$

$f'(c)=-\frac{2}{c^3}$

From $-\frac{2}{c^3}=-\frac{a+b}{a^2b^2}$, $\quad c=\sqrt[3]{\frac{2a^2b^2}{a+b}}$

3. Given a function $f(x)$ such that $f\left(-\frac{1}{2}\right)=-3$ and $f'(x)\le 1$ for all values of x, determine the maximum value of $f(0)$.

[Sol] Using the Mean Value Theorem over the interval $\left[-\frac{1}{2},0\right]$,

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$
$$f(b)=f(a)+f'(c)\cdot(b-a)$$
$$f(0)=f\left(-\frac{1}{2}\right)+f'(c)\cdot\frac{1}{2}$$
$$=-3+\frac{1}{2}f'(c)$$

Since $f'(x)\le 1$ for all values of x,
the maximum value of $f(0)$ is $-3+\frac{1}{2}=-\frac{5}{2}$

Indefinite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	1	2-3	4-5	6~

$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.
All antiderivatives are called the *Indefinite Integral* of $f(x)$.
Therefore, the Indefinite Integral of $f(x)$ is $F(x)$, and this can be expressed as:

$$\int f(x)dx = F(x) + C \qquad \text{(where } C \text{ is the constant of integration)}$$

Complete and check your answers.
Given that $F(x) = 3x^6 + 2x^3 \quad \ldots①$ is an antiderivative of a function $f(x)$, show that $\int f(x)dx = F(x) + C$.

[Sol] Differentiating $F(x)$ with respect to x,

$$F'(x) = \boxed{18x^5 + 6x^2}$$

Since $F(x)$ is an antiderivative of a function $f(x)$, $F'(x) = f(x)$.

$$\text{Therefore,} \quad f(x) = \boxed{18x^5 + 6x^2}$$

$$\int f(x)dx = \int \boxed{18x^5 + 6x^2}\, dx = \boxed{3x^6 + 2x^3} + C$$

From ①

$$\therefore \int f(x)dx = F(x) + C$$

Answers: $18x^5 + 6x^2$, $18x^5 + 6x^2$, $18x^5 + 6x^2$, $3x^6 + 2x^3$

1. Evaluate each of the following derivatives.

(1) $(2x^3 + x^4)' = 6x^2 + 4x^3$

(2) $(3\sqrt{x})' = \dfrac{3}{2\sqrt{x}}$

(3) $\left(\dfrac{1}{n+1}x^{n+1}\right)' = x^n$

(4) $(\ln|x|)' = \dfrac{1}{x}$

(5) $(e^x)' = e^x$

(6) $\left(\dfrac{a^x}{\ln a}\right)' = a^x$

2. Using the results of exercise 1., and letting C be the constant of integration, evaluate each of the following indefinite integrals.

(1) $\int (6x^2 + 4x^3)dx = 2x^3 + x^4 + C$

(2) $\int \dfrac{3}{2\sqrt{x}}dx = 3\sqrt{x} + C$

(3) $\int x^n dx = \dfrac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$

(4) $\int \dfrac{dx}{x} = \ln|x| + C$

(5) $\int e^x dx = e^x + C$

(6) $\int a^x dx = \dfrac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$

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Formulas

Formulas	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$,	$\int \frac{dx}{x} = \ln\lvert x\rvert + C$
	$\int e^x\,dx = e^x + C$,	$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a>0, a\neq 1)$

3. In each exercise, evaluate the given integral, and then check your answer through differentiation.

(1) $\int \sqrt[3]{x^2}\,dx = \int x^{\boxed{1}}\,dx = \frac{3}{5}x^{\frac{5}{3}} + C = \frac{3}{5}x\sqrt[3]{x^2} + C$

Checking the answer. $\left(\frac{3}{5}x\sqrt[3]{x^2} + C\right)' = \left(\frac{3}{5}x^{\frac{5}{3}} + C\right)' = x^{\frac{2}{3}} = \sqrt[3]{x^2}$

(2) $\int \frac{dx}{x\sqrt{x}} = \int x^{-\frac{3}{2}}\,dx = -2x^{-\frac{1}{2}} + C = -\frac{2}{\sqrt{x}} + C$

Checking the answer. $\left(-\frac{2}{\sqrt{x}} + C\right)' = (-2x^{-\frac{1}{2}} + C)' = x^{-\frac{3}{2}} = \frac{1}{x\sqrt{x}}$

(3) $\int (x+2)^4\,dx = \frac{1}{5}(x+2)^5 + C$

Checking the answer. $\left[\frac{1}{5}(x+2)^5 + C\right]' = (x+2)^4$

(4) $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$

Checking the answer. $\left(\frac{1}{2}e^{2x} + C\right)' = e^{2x}$

(5) $\int \left(\frac{1}{3}\right)^x dx = \dfrac{\left(\frac{1}{3}\right)^x}{\ln\frac{1}{3}} + C$

Checking the answer. $\left[\dfrac{\left(\frac{1}{3}\right)^x}{\ln\frac{1}{3}} + C\right]' = \left(\frac{1}{3}\right)^x$

Indefinite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	1	2–3	4	5–

1. Evaluate each of the following indefinite integrals.

(1) $\int \sqrt{x-3}\,dx = \frac{2}{3}(x-3)^{\frac{3}{2}} + C = \frac{2}{3}(x-3)\sqrt{x-3} + C$

(2) $\int \frac{dx}{(x+2)^3} = -\frac{1}{2}(x+2)^{-2} + C = -\frac{1}{2(x+2)^2} + C$

(3) $\int (2x-3)^2 dx = \frac{1}{6}(2x-3)^3 + C$

(4) $\int \frac{dx}{2x+1} = \frac{1}{2}\ln|2x+1| + C$

(5) $\int e^{-x} dx = -e^{-x} + C$

(6) $\int 2^{x-3} dx = \frac{2^{x-3}}{\ln 2} + C$

(7) $\int \frac{2}{\sqrt[3]{x-4}} dx = 2\int (x-4)^{-\frac{1}{3}} dx$

$= 3(x-4)^{\frac{2}{3}} + C$

$= 3\sqrt[3]{(x-4)^2} + C$

O 72 b

2. Evaluate each of the following derivatives.

(1) $\left(\frac{1}{2}e^{2x+3}\right)' = e^{2x+3}$

(2) $\left(-\frac{1}{2}\ln|1-2x|\right)' = \frac{1}{1-2x}$

(3) $\left[\frac{1}{5}(3x+2)^{\frac{5}{3}}\right]' = (3x+2)^{\frac{2}{3}}$

(4) $\left(\frac{3^{2x-1}}{2\ln 3}\right)' = 3^{2x-1}$

3. Using the results of exercise 2., and letting C be the constant of integration, evaluate each of the following indefinite integrals.

(1) $\int e^{2x+3}dx = \frac{1}{2}e^{2x+3} + C$

(2) $\int \frac{dx}{1-2x} = -\frac{1}{2}\ln|1-2x| + C$

(3) $\int (3x+2)^{\frac{2}{3}}dx = \frac{1}{5}(3x+2)^{\frac{5}{3}} + C$

(4) $\int 3^{2x-1}\,dx = \frac{3^{2x-1}}{2\ln 3} + C$

Formula

> Given that $f(x) = F'(x)$,
>
> $$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C \quad (a \neq 0)$$

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O 73 a

Indefinite Integrals 1

Time ___ : ___ to ___ : ___ Date ______ Name ______

100%	90%	80%	70%	69%~
(mistakes) 0	-	1	2	3~

Evaluate each of the following indefinite integrals.

(1) $\displaystyle\int \frac{x^2-2x-3}{x^2}\,dx = \int\left(1-\frac{2}{\boxed{x}}-\boxed{\frac{3}{x^2}}\right)dx$

$\displaystyle = x-2\ln|x|+\frac{3}{x}+C$

(2) $\displaystyle\int \frac{\sqrt{x}+\sqrt[3]{x}-1}{x}\,dx = \int\left(x^{-\frac{1}{2}}+x^{-\frac{2}{3}}-\frac{1}{x}\right)dx$

$\displaystyle = 2x^{\frac{1}{2}}+3x^{\frac{1}{3}}-\ln x+C$

$\displaystyle = 2\sqrt{x}+3\sqrt[3]{x}-\ln x+C$

(3) $\displaystyle\int \frac{(x^2+1)^2}{3x^2}\,dx = \frac{1}{3}\int \frac{x^4+2x^2+1}{x^2}\,dx$

$\displaystyle = \frac{1}{3}\int\left(x^2+2+\frac{1}{x^2}\right)dx$

$\displaystyle = \frac{1}{3}\left(\frac{1}{3}x^3+2x-\frac{1}{x}\right)+C$

$\displaystyle = \frac{1}{9}x^3+\frac{2}{3}x-\frac{1}{3x}+C$

(4) $\displaystyle\int \frac{(2x-1)^3}{x^5}\,dx = \int \frac{8x^3-12x^2+6x-1}{x^5}\,dx$

$\displaystyle = \int\left(\frac{8}{x^2}-\frac{12}{x^3}+\frac{6}{x^4}-\frac{1}{x^5}\right)dx$

$\displaystyle = -\frac{8}{x}+\frac{6}{x^2}-\frac{2}{x^3}+\frac{1}{4x^4}+C$

O 73 b

(5) $\displaystyle\int \frac{x^2+5x+12}{\sqrt{x}}dx = \int (x^{\frac{3}{2}}+5x^{\frac{1}{2}}+12x^{-\frac{1}{2}})dx$

$$= \frac{2}{5}x^{\frac{5}{2}}+\frac{10}{3}x^{\frac{3}{2}}+24x^{\frac{1}{2}}+C$$

$$= \frac{2}{5}x^2\sqrt{x}+\frac{10}{3}x\sqrt{x}+24\sqrt{x}+C$$

(6) $\displaystyle\int \frac{x+\sqrt{x}}{\sqrt[3]{x}}dx = \int (x^{\frac{2}{3}}+x^{\frac{1}{6}})dx$

$$= \frac{3}{5}x^{\frac{5}{3}}+\frac{6}{7}x^{\frac{7}{6}}+C$$

$$= \frac{3}{5}x\sqrt[3]{x^2}+\frac{6}{7}x\sqrt[6]{x}+C$$

(7) $\displaystyle\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2dx = \int \left(x+2+\frac{1}{x}\right)dx$

$$= \frac{1}{2}x^2+2x+\ln x+C$$

(8) $\displaystyle\int \frac{(\sqrt{x}+1)^3}{\sqrt{x}}dx = \int \frac{x\sqrt{x}+3x+3\sqrt{x}+1}{\sqrt{x}}dx$

$$= \int (x+3x^{\frac{1}{2}}+3+x^{-\frac{1}{2}})dx$$

$$= \frac{1}{2}x^2+2x^{\frac{3}{2}}+3x+2x^{\frac{1}{2}}+C$$

$$= \frac{1}{2}x^2+2x\sqrt{x}+3x+2\sqrt{x}+C$$

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O 74 a

Indefinite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	2	3–

Evaluate each of the following indefinite integrals.

(1) $\displaystyle\int\frac{x-\sqrt[3]{x}}{\sqrt{x}}dx=\int(x^{\frac{1}{2}}-x^{\boxed{-\frac{1}{6}}})dx$

$\displaystyle=\frac{2}{3}x^{\frac{3}{2}}-\frac{6}{5}x^{\frac{5}{6}}+C$

$\displaystyle=\frac{2}{3}x\sqrt{x}-\frac{6}{5}\sqrt[6]{x^5}+C$

(2) $\displaystyle\int\frac{dx}{(3-4x)^2}=\int(3-4x)^{-2}dx$

$\displaystyle=-\frac{1}{-4(3-4x)}+C=\frac{1}{4(3-4x)}+C$

(3) $\displaystyle\int\frac{dx}{2-3x}=-\frac{1}{3}\ln|2-3x|+C$

(4) $\displaystyle\int\left(\frac{e^x+e^{-x}}{2}\right)^2dx=\frac{1}{4}\int(e^{2x}+2+e^{-2x})dx$

$\displaystyle=\frac{1}{4}\left(\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}\right)+C$

$\displaystyle=\frac{1}{8}e^{2x}+\frac{1}{2}x-\frac{1}{8}e^{-2x}+C$

(5) $\displaystyle\int\frac{e^{2x}-1}{e^x-1}dx=\int\frac{(e^x-1)(\boxed{e^x+1})}{e^x-1}dx$

$\displaystyle=\int(e^x+1)dx$

$=e^x+x+C$

O 74 b

(6) $$\int \frac{dx}{\sqrt{x+2}+\sqrt{x}} = \int \frac{\boxed{\sqrt{x+2}-\sqrt{x}}}{\left(\sqrt{x+2}+\sqrt{x}\right)\boxed{\left(\sqrt{x+2}-\sqrt{x}\right)}}dx$$
$$= \frac{1}{2}\int \left[(x+2)^{\frac{1}{2}} - x^{\frac{1}{2}}\right]dx$$
$$= \frac{1}{2}\left[\frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right] + C$$
$$= \frac{1}{3}(x+2)\sqrt{x+2} - \frac{1}{3}x\sqrt{x} + C$$

(7) $$\int \frac{x}{\sqrt{x+1}+1}dx = \int \frac{x\left(\sqrt{x+1}-1\right)}{\left(\sqrt{x+1}+1\right)\left(\sqrt{x+1}-1\right)}dx$$
$$= \int \left(\sqrt{x+1}-1\right)dx$$
$$= \frac{2}{3}(x+1)^{\frac{3}{2}} - x + C$$
$$= \frac{2}{3}(x+1)\sqrt{x+1} - x + C$$

(8) $$\int \frac{dx}{\sqrt{2-x}-\sqrt{1-x}} = \int \frac{\sqrt{2-x}+\sqrt{1-x}}{\left(\sqrt{2-x}-\sqrt{1-x}\right)\left(\sqrt{2-x}+\sqrt{1-x}\right)}dx$$
$$= \int \left(\sqrt{2-x}+\sqrt{1-x}\right)dx$$
$$= -\frac{2}{3}(2-x)^{\frac{3}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$$
$$= -\frac{2}{3}(2-x)\sqrt{2-x} - \frac{2}{3}(1-x)\sqrt{1-x} + C$$

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O 75 a Indefinite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69% -
(mistakes) 0	-	1	2	3 -

Evaluate each of the following indefinite integrals.

Ex. $\displaystyle\int\frac{2x+1}{x-1}dx=\int\frac{2(x-1)+3}{x-1}dx$

$\displaystyle=\int\left(2+\frac{3}{x-1}\right)dx$

$=2x+3\ln|x-1|+C$

(1) $\displaystyle\int\frac{x^3+2}{x-1}dx=\int\left(x^2+x+1+\frac{3}{\boxed{x-1}}\right)dx$

$\displaystyle=\frac{x^3}{3}+\frac{x^2}{2}+x+3\ln|x-1|+C$

(2) $\displaystyle\int\frac{dx}{(x+1)(x+2)}=\int\left(\frac{1}{x+1}-\frac{1}{\boxed{x+2}}\right)dx$

$=\ln|x+1|-\ln|x+2|+C$

$\displaystyle\left[=\ln\left|\frac{x+1}{x+2}\right|+C\right]$

(3) $\displaystyle\int\frac{2}{x^2-1}dx=\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$

$=\ln|x-1|-\ln|x+1|+C$

$\displaystyle\left[=\ln\left|\frac{x-1}{x+1}\right|+C\right]$

O 75 b

(4) $\displaystyle\int\frac{(x+1)^2}{x-1}dx=\int\left(x+3+\frac{4}{x-1}\right)dx$

$\displaystyle=\frac{1}{2}x^2+3x+4\ln|x-1|+C$

(5) $\displaystyle\int\left(\frac{e^{2x}+e^{-2x}}{2}\right)^2dx=\int\frac{e^{4x}+2+e^{-4x}}{4}dx$

$\displaystyle=\frac{1}{4}\left(\frac{1}{4}e^{4x}+2x-\frac{1}{4}e^{-4x}\right)+C$

$\displaystyle=\frac{1}{16}e^{4x}+\frac{1}{2}x-\frac{1}{16}e^{-4x}+C$

(6) $\displaystyle\int\frac{dx}{4x^2-1}=\int\frac{dx}{(2x+1)(2x-1)}$

$\displaystyle=\frac{1}{2}\int\left(\frac{1}{2x-1}-\frac{1}{2x+1}\right)dx$

$\displaystyle=\frac{1}{2}\left(\frac{1}{2}\ln|2x-1|-\frac{1}{2}\ln|2x+1|\right)+C$

$\displaystyle=\frac{1}{4}\ln|2x-1|-\frac{1}{4}\ln|2x+1|+C\left[=\frac{1}{4}\ln\left|\frac{2x-1}{2x+1}\right|+C\right]$

(7) $\displaystyle\int\frac{x}{\sqrt{x+2}}dx=\int\frac{(x+2)-2}{\sqrt{x+2}}dx$

$\displaystyle=\int\left(\sqrt{x+2}-\frac{2}{\sqrt{x+2}}\right)dx$

$\displaystyle=\frac{2}{3}(x+2)^{\frac{3}{2}}-4(x+2)^{\frac{1}{2}}+C$

$\displaystyle=\frac{2}{3}(x+2)\sqrt{x+2}-4\sqrt{x+2}+C\left[=\frac{2}{3}(x-4)\sqrt{x+2}+C\right]$

O 76 a Indefinite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	1	2-3	4	5-

1. Evaluate each of the following derivatives.

(1) $(\sin x)' = \cos x$

(2) $(\cos x)' = -\sin x$

(3) $(\tan x)' = \dfrac{1}{\cos^2 x}$

(4) $\left(\dfrac{1}{\tan x}\right)' = \dfrac{-\dfrac{1}{\cos^2 x}}{\tan^2 x} = -\dfrac{1}{\sin^2 x}$

2. Using the results of exercise 1., and letting C be the constant of integration, evaluate each of the following indefinite integrals.

(1) $\displaystyle\int \sin x\,dx = -\cos x + C$

(2) $\displaystyle\int \cos x\,dx = \sin x + C$

(3) $\displaystyle\int \frac{dx}{\cos^2 x} = \tan x + C$

(4) $\displaystyle\int \frac{dx}{\sin^2 x} = -\frac{1}{\tan x} + C$

(5) $\displaystyle\int \tan^2 x\,dx = \int\left(\frac{\sin^2 x}{\cos^2 x}\right)dx = \int\left(\frac{1}{\cos^2 x} - \boxed{1}\right)dx$

$= \tan x - x + C$

(6) $\displaystyle\int \frac{dx}{\tan^2 x} = \int\left(\frac{\cos^2 x}{\boxed{\sin^2 x}}\right)dx = \int\left(\frac{1}{\sin^2 x} - 1\right)dx$

$\displaystyle= -\frac{1}{\tan x} - x + C$

O 76 b

Formulas

$$\int \sin x\,dx = -\cos x + C, \qquad \int \cos x\,dx = \sin x + C$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C, \qquad \int \frac{dx}{\sin^2 x} = -\frac{1}{\tan x} + C$$

3. Evaluate each of the following indefinite integrals.

(1) $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

(2) $\int \cos 3x\,dx = \frac{1}{3}\sin 3x + C$

(3) $\int \cos x(2 - \tan x)\,dx = \int (2\cos x - \sin x)\,dx$

$$= 2\sin x + \cos x + C$$

(4) $\int \frac{\cos^2 x}{1 + \sin x}\,dx = \int \frac{1 - \sin^2 x}{1 + \sin x}\,dx$

$$= \int \frac{(1 + \sin x)(1 - \sin x)}{1 + \sin x}\,dx$$

$$= \int (1 - \sin x)\,dx$$

$$= x + \cos x + C$$

(5) $\int \frac{1 - \sin^3 x}{\sin^2 x}\,dx = \int \left(\frac{1}{\sin^2 x} - \sin x\right)dx$

$$= -\frac{1}{\tan x} + \cos x + C$$

Indefinite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	2	3–

1. Fill in the blank boxes to review the following trigonometric formulas.

$$\sin 2\theta = 2(\sin\theta)(\boxed{\cos\theta})$$

$$\cos 2\theta = \cos^2\theta - \boxed{\sin^2\theta} = 2\cos^2\theta - \boxed{1} = 1 - 2\boxed{\sin^2\theta}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\boxed{\alpha-\beta})]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\boxed{\alpha-\beta})]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \boxed{\cos}(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = -\frac{1}{2}[\boxed{\cos(\alpha+\beta)} - \cos(\alpha-\beta)]$$

2. Using the above formulas, evaluate each of the following indefinite integrals.

(1) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

(2) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

(3) $$\int \sin x\cos x\,dx = \frac{1}{2}\int \sin 2x\,dx$$
$$= -\frac{1}{4}\cos 2x + C$$

O 77 b

(4) $$\int (\sin x+\cos x)^2 dx=\int (\sin^2 x+2\sin x\cos x+\cos^2 x)dx$$
$$=\int (1+\sin 2x)dx$$
$$=x-\frac{1}{2}\cos 2x+C$$

(5) $$\int \sin 3x\cos x dx=\frac{1}{2}\int (\sin 4x+\sin 2x)dx$$
$$=\frac{1}{2}\left(-\frac{1}{4}\cos 4x-\frac{1}{2}\cos 2x\right)+C$$
$$=-\frac{1}{8}\cos 4x-\frac{1}{4}\cos 2x+C$$

(6) $$\int \sin 3x\sin 2x dx=-\frac{1}{2}\int (\cos 5x-\cos x)dx$$
$$=-\frac{1}{2}\left(\frac{1}{5}\sin 5x-\sin x\right)+C$$
$$=-\frac{1}{10}\sin 5x+\frac{1}{2}\sin x+C$$

(7) $$\int \cos 2x\cos 3x dx=\frac{1}{2}\int (\cos 5x+\cos x)dx$$
$$=\frac{1}{2}\left(\frac{1}{5}\sin 5x+\sin x\right)+C$$
$$=\frac{1}{10}\sin 5x+\frac{1}{2}\sin x+C$$

Indefinite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	1	–	2–

Evaluate each of the following indefinite integrals.

(1) $\int \cos^2 \frac{x}{2} dx = \int \frac{1 + \cos x}{2} dx$

$= \frac{1}{2}x + \frac{1}{2}\sin x + C$

(2) $\int \sin x(1 + \cos x) dx = \int \left(\sin x + \frac{1}{2}\sin 2x\right) dx$

$= -\cos x - \frac{1}{4}\cos 2x + C$

(3) $\int \left(\sin x + \frac{1}{\sin x}\right)^2 dx = \int \left(\sin^2 x + 2 + \frac{1}{\sin^2 x}\right) dx$

$= \int \left(\frac{1 - \cos 2x}{2} + 2 + \frac{1}{\sin^2 x}\right) dx$

$= \int \left(\frac{5}{2} - \frac{1}{2}\cos 2x + \frac{1}{\sin^2 x}\right) dx$

$= \frac{5}{2}x - \frac{1}{4}\sin 2x - \frac{1}{\tan x} + C$

O 78 b

(4) $$\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2dx=\int\left(\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}\right)dx$$
$$=\int(1+\sin x)dx$$
$$=x-\cos x+C$$

(5) $$\int\left(\tan x+\frac{1}{\tan x}\right)^2dx=\int\left(\tan^2x+2+\frac{1}{\tan^2x}\right)dx$$
$$=\int\left[\left(\frac{1}{\cos^2x}-1\right)+2+\left(\frac{1}{\sin^2x}-1\right)\right]dx$$
$$=\int\left(\frac{1}{\cos^2x}+\frac{1}{\sin^2x}\right)dx$$
$$=\tan x-\frac{1}{\tan x}+C$$

(6) $$\int\frac{\cos 2x}{\sin x-\cos x}dx=\int\frac{\cos^2x-\sin^2x}{\sin x-\cos x}dx$$
$$=\int\frac{(\cos x-\sin x)(\cos x+\sin x)}{\sin x-\cos x}dx$$
$$=-\int(\cos x+\sin x)dx$$
$$=-(\sin x-\cos x)+C$$
$$=\cos x-\sin x+C$$

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Indefinite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

Evaluate each of the following indefinite integrals.

(1) $\int \sin^3 x dx = \frac{1}{4}\int(3\sin x - \sin 3x)dx$ (**Hint:** $\sin 3x = 3\sin x - 4\sin^3 x$)

$= \frac{1}{4}\left(-3\cos x + \frac{1}{3}\cos 3x\right) + C$

$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$

(2) $\int \cos^3 x dx = \frac{1}{4}\int(3\cos x + \cos 3x)dx$

$= \frac{1}{4}\left(3\sin x + \frac{1}{3}\sin 3x\right) + C$

$= \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$

(3) $\int \sin^4 x dx = \int(\sin^2 x)^2 dx$

$= \int\left(\frac{1-\cos 2x}{2}\right)^2 dx$

$= \frac{1}{4}\int(1 - 2\cos 2x + \cos^2 2x)dx$

$= \frac{1}{4}\int\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx$

$= \frac{1}{8}\int(3 - 4\cos 2x + \cos 4x)dx$

$= \frac{1}{8}\left(3x - 2\sin 2x + \frac{1}{4}\sin 4x\right) + C$

$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

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(4) $\displaystyle\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{dx}{(\sin x \cos x)^2}$

$$= \int \frac{dx}{\left(\frac{1}{2}\sin 2x\right)^2}$$

$$= 4\int \frac{dx}{\sin^2 2x}$$

$$= 4\cdot\left(-\frac{1}{2}\right)\frac{1}{\tan 2x} + C$$

$$= -\frac{2}{\tan 2x} + C \quad \left[= \frac{\tan^2 x - 1}{\tan x} + C\right]$$

(5) $\displaystyle\int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\frac{x}{2}}$

$$= \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2}}$$

$$= \frac{1}{2}\cdot 2\tan\frac{x}{2} + C$$

$$= \tan\frac{x}{2} + C$$

(6) $\displaystyle\int (\sqrt{3}\sin x + \cos x)^2 dx = \int (3\sin^2 x + 2\sqrt{3}\sin x\cos x + \cos^2 x)dx$

$$= \int\left(3\cdot\frac{1-\cos 2x}{2} + \sqrt{3}\sin 2x + \frac{1+\cos 2x}{2}\right)dx$$

$$= \int (2 - \cos 2x + \sqrt{3}\sin 2x)dx$$

$$= 2x - \frac{1}{2}\sin 2x - \frac{\sqrt{3}}{2}\cos 2x + C$$

(Alternate Solution

$$\int (\sqrt{3}\sin x + \cos x)^2 dx = \int\left[2\sin\left(x+\frac{\pi}{6}\right)\right]^2 dx = \int 4\cdot\frac{1-\cos\left(2x+\frac{\pi}{3}\right)}{2}dx$$

$$= 2x - \sin\left(2x+\frac{\pi}{3}\right) + C$$
)

Indefinite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	2	3–

Evaluate each of the following indefinite integrals.

(1) $\displaystyle\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}dx = \int \frac{x\sqrt{x}+6x+12\sqrt{x}+8}{\sqrt{x}}dx$

$\displaystyle= \int\left(x+6\sqrt{x}+12+\frac{8}{\sqrt{x}}\right)dx$

$\displaystyle= \frac{1}{2}x^2+4x^{\frac{3}{2}}+12x+16x^{\frac{1}{2}}+C$

$\displaystyle= \frac{1}{2}x^2+4x\sqrt{x}+12x+16\sqrt{x}+C$

(2) $\displaystyle\int \frac{1-e^{2x}}{1+e^x}dx = \int \frac{(1+e^x)(1-e^x)}{1+e^x}dx$

$\displaystyle= \int(1-e^x)dx$

$= x-e^x+C$

(3) $\displaystyle\int \frac{2x-1}{2x+1}dx = \int\left(1-\frac{2}{2x+1}\right)dx$

$\displaystyle= x-2\cdot\frac{1}{2}\ln|2x+1|+C$

$= x-\ln|2x+1|+C$

(4) $\displaystyle\int \frac{dx}{\sqrt{x+2}-\sqrt{x+1}} = \int\left(\sqrt{x+2}+\sqrt{x+1}\right)dx$

$\displaystyle= \frac{2}{3}(x+2)^{\frac{3}{2}}+\frac{2}{3}(x+1)^{\frac{3}{2}}+C$

$\displaystyle= \frac{2}{3}(x+2)\sqrt{x+2}+\frac{2}{3}(x+1)\sqrt{x+1}+C$

O 80 b

(5) $\displaystyle\int \frac{1+\cos^3 x}{\cos^2 x}\,dx = \int\left(\frac{1}{\cos^2 x} + \cos x\right)dx$

$\displaystyle = \tan x + \sin x + C$

(6) $\displaystyle\int (1-\cos x)^2 dx = \int (1 - 2\cos x + \cos^2 x)\,dx$

$\displaystyle = \int\left(1 - 2\cos x + \frac{1+\cos 2x}{2}\right)dx$

$\displaystyle = \int\left(\frac{3}{2} - 2\cos x + \frac{1}{2}\cos 2x\right)dx$

$\displaystyle = \frac{3}{2}x - 2\sin x + \frac{1}{4}\sin 2x + C$

(7) $\displaystyle\int \sin 3x \cos 6x\,dx = \frac{1}{2}\int (\sin 9x - \sin 3x)\,dx$

$\displaystyle = \frac{1}{2}\left(-\frac{1}{9}\cos 9x + \frac{1}{3}\cos 3x\right) + C$

$\displaystyle = -\frac{1}{18}\cos 9x + \frac{1}{6}\cos 3x + C$

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	–	–	–

Ex. Evaluate the following indefinite integral.

$$\int (2x-5)^3 dx$$

[Sol] Letting $t = 2x - 5$, …①

Solving ① for x, $x = \dfrac{t+5}{2}$ …②

Differentiating ② with respect to t,

$$\frac{dx}{dt} = \frac{1}{2}$$

Solving for dx, $dx = \dfrac{1}{2}dt$ …③

Using ① and ③, we can express the given integral in terms of t:

$$\int (2x-5)^3 dx = \int t^3 \cdot \frac{1}{2} dt = \frac{t^4}{8} + C$$

Expressing this value in terms of the original variable x,

$$\int (2x-5)^3 dx = \frac{1}{8}(2x-5)^4 + C$$

The method shown in the above example is called *Integration by Substitution.*

Evaluate each of the following indefinite integrals by applying the method of substitution.

(1) $\int (3x+2)^5 dx$

[Sol] Letting $t = 3x + 2$, …①

Solving ① for x, $x = \dfrac{t-2}{3}$ …②

Differentiating ② with respect to t,

$$\frac{dx}{dt} = \frac{1}{3}$$

Solving for dx, $dx = \dfrac{1}{3}dt$ …③

Using ① and ③, we can express the given integral in terms of t:

$$\int (3x+2)^5 dx = \int t^5 \cdot \frac{1}{3} dt = \frac{1}{18}t^6 + C$$

Expressing this value in terms of the original variable x,

$$\int (3x+2)^5 dx = \frac{1}{18}(3x+2)^6 + C$$

O 81 b

(2) $\displaystyle\int \frac{dx}{3x-1}$

[Sol] Letting $t = 3x - 1$,

$$x = \frac{t+1}{3}, \qquad \therefore dx = \frac{1}{3}dt$$

Therefore, $\displaystyle\int \frac{1}{3x-1}dx = \int \frac{1}{t} \cdot \frac{1}{3}dt$

$$= \frac{1}{3}\ln|t| + C$$

$$= \frac{1}{3}\ln|3x-1| + C$$

(3) $\displaystyle\int x(3x+2)^3 dx$

[Sol] Letting $t = 3x + 2$,

$$x = \frac{t-2}{3}, \quad \therefore dx = \frac{1}{3}dt$$

Therefore, $\displaystyle\int x(3x+2)^3 dx = \int \frac{t-2}{3} \cdot t^3 \cdot \frac{1}{3}dt$

$$= \frac{1}{9}\int (t^4 - 2t^3)dt$$

$$= \frac{1}{45}t^5 - \frac{1}{18}t^4 + C$$

$$= \frac{1}{45}(3x+2)^5 - \frac{1}{18}(3x+2)^4 + C$$

$$\left[= \frac{1}{90}(3x+2)^4(6x-1) + C\right]$$

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	1	-	2~

Evaluate each of the following indefinite integrals by applying the method of substitution.

(1) $\int (x^3+1)^2 \underline{x^2 dx}$

[Sol] Letting $t = x^3+1$,

$$\frac{dt}{dx} = \boxed{3x^2}$$

$$\boxed{3x^2}\,dx = dt$$

$$\underline{x^2 dx} = \boxed{\frac{1}{3}dt}$$

$$\therefore \int (x^3+1)^2 x^2 dx = \int \boxed{t^2 \cdot \frac{1}{3}}\,dt$$

$$= \frac{1}{9}t^3 + C = \frac{1}{9}(x^3+1)^3 + C$$

$$\left[= \frac{1}{9}x^9 + \frac{1}{3}x^6 + \frac{1}{3}x^3 + C\right]$$

(2) $\int \frac{x}{x^2+1}dx$

[Sol] Letting $t = x^2+1$,

$$\frac{dt}{dx} = 2x$$

$$2x\,dx = dt$$

$$x\,dx = \frac{1}{2}dt$$

$$\therefore \int \frac{x}{x^2+1}dx = \int \frac{1}{t}\cdot\frac{1}{2}dt$$

$$= \frac{1}{2}\ln|t| + C$$

$$= \frac{1}{2}\ln(x^2+1) + C$$

O 82 b

(3) $\int \frac{x}{(x^2+4)^2} dx$

[Sol] Letting $t = x^2 + 4$,

$$\frac{dt}{dx} = 2x, \text{ therefore, } 2xdx = dt \text{ and } xdx = \frac{1}{2}dt$$

$$\therefore \int \frac{x}{(x^2+4)^2} dx = \frac{1}{2}\int \frac{dt}{t^2}$$
$$= -\frac{1}{2t} + C$$
$$= -\frac{1}{2(x^2+4)} + C$$

(4) $\int \frac{x}{(1+x^2)^3} dx$

[Sol] Letting $t = 1 + x^2$,

$$\frac{dt}{dx} = 2x, \text{ therefore, } 2xdx = dt \text{ and } xdx = \frac{1}{2}dt$$

$$\therefore \int \frac{x}{(1+x^2)^3} dx = \int \frac{1}{t^3}\cdot\frac{1}{2}dt$$
$$= -\frac{1}{4}t^{-2} + C$$
$$= -\frac{1}{4(1+x^2)^2} + C$$

(5) $\int \frac{2x^2}{x^3+1} dx$

[Sol] Letting $t = x^3 + 1$,

$$\frac{dt}{dx} = 3x^2, \text{ therefore, } 3x^2 dx = dt \text{ and } x^2 dx = \frac{1}{3}dt$$

$$\therefore \int \frac{2x^2}{x^3+1} dx = \int \frac{2}{t}\cdot\frac{1}{3}dt$$
$$= \frac{2}{3}\ln|t| + C$$
$$= \frac{2}{3}\ln|x^3+1| + C$$

KUMON

O 83 a

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Evaluate each of the following indefinite integrals by applying the method of substitution.

(1) $\displaystyle\int \frac{x}{(x-1)^3}\,dx$

[Sol] Letting $t = x - 1$, $x = \boxed{t+1}$

$$\frac{dx}{dt} = \boxed{1}$$

$$dx = \boxed{dt}$$

$$\therefore \int \frac{x}{(x-1)^3}\,dx = \int \boxed{\frac{t+1}{t^3}}\,dt$$

$$= \int \left(\frac{1}{t^2} + \frac{1}{t^3}\right) dt$$

$$= -\frac{1}{t} - \frac{1}{2t^2} + C$$

$$= -\frac{2t+1}{2t^2} + C$$

$$= -\frac{2(x-1)+1}{2(x-1)^2} + C = -\frac{2x-1}{2(x-1)^2} + C$$

(2) $\displaystyle\int \frac{x+1}{(3x-1)^3}\,dx$

[Sol] Letting $t = 3x - 1$, $x = \dfrac{t+1}{3}$

$$\frac{dx}{dt} = \frac{1}{3}, \text{ therefore, } dx = \frac{1}{3}dt$$

$$\therefore \int \frac{x+1}{(3x-1)^3}\,dx = \int \left(\frac{t+1}{3} + 1\right) \cdot \frac{1}{t^3} \cdot \frac{1}{3}\,dt$$

$$= \frac{1}{9} \int \frac{t+4}{t^3}\,dt$$

$$= \frac{1}{9} \int \left(\frac{1}{t^2} + \frac{4}{t^3}\right) dt$$

$$= \frac{1}{9}\left(-\frac{1}{t} - \frac{2}{t^2}\right) + C$$

$$= -\frac{t+2}{9t^2} + C = -\frac{3x+1}{9(3x-1)^2} + C$$

O 83 b

(3) $\int x\sqrt{2x-1}\,dx$

[Sol] Letting $t=2x-1$, $x=\dfrac{t+1}{2}$

$$\frac{dx}{dt}=\frac{1}{2}, \text{ therefore, } dx=\frac{1}{2}dt$$

$$\begin{aligned}
\therefore \int x\sqrt{2x-1}\,dx &= \int \frac{t+1}{2}\sqrt{t}\cdot\frac{1}{2}dt \\
&= \frac{1}{4}\int (t\sqrt{t}+\sqrt{t})dt \\
&= \frac{1}{4}\left(\frac{2}{5}t^{\frac{5}{2}}+\frac{2}{3}t^{\frac{3}{2}}\right)+C \\
&= \frac{1}{30}t^{\frac{3}{2}}(3t+5)+C \\
&= \frac{1}{15}(2x-1)(3x+1)\sqrt{2x-1}+C
\end{aligned}$$

(4) $\int \dfrac{3x-1}{\sqrt{x+1}}dx$

[Sol] Letting $t=x+1$, $x=t-1$

$$\frac{dx}{dt}=1, \text{ therefore, } dx=dt$$

$$\begin{aligned}
\therefore \int \frac{3x-1}{\sqrt{x+1}}dx &= \int \frac{3(t-1)-1}{\sqrt{t}}dt \\
&= \int\left(\frac{3t}{t^{\frac{1}{2}}}-\frac{4}{t^{\frac{1}{2}}}\right)dt \\
&= \int (3t^{\frac{1}{2}}-4t^{-\frac{1}{2}})dt \\
&= 2t^{\frac{3}{2}}-8t^{\frac{1}{2}}+C \\
&= 2t^{\frac{1}{2}}(t-4)+C \\
&= 2(x+1)^{\frac{1}{2}}\{(x+1)-4\}+C \\
&= 2(x-3)\sqrt{x+1}+C
\end{aligned}$$

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Evaluate each of the following indefinite integrals by applying the method of substitution.

(1) $\displaystyle\int \frac{x^2}{\sqrt{x+2}}\,dx$

[Sol] Letting $t=\sqrt{x+2}$, $x=t^2-2$

$$\frac{dx}{dt}=2t, \text{ therefore, } dx=2t\,dt$$

$$\therefore \int \frac{x^2}{\sqrt{x+2}}\,dx=\int \frac{(t^2-2)^2}{t}\,2t\,dt$$

$$=2\int (t^4-4t^2+4)\,dt$$

$$=2\left(\frac{1}{5}t^5-\frac{4}{3}t^3+4t\right)+C$$

$$=\frac{2}{15}t(3t^4-20t^2+60)+C$$

$$=\frac{2}{15}(3x^2-8x+32)\sqrt{x+2}+C$$

(2) $\displaystyle\int \frac{x^2}{(x+1)\sqrt{x+1}}\,dx$

[Sol] Letting $t=\sqrt{x+1}$, $x=t^2-1$

$$\frac{dx}{dt}=2t, \text{ therefore, } dx=2t\,dt$$

$$\therefore \int \frac{x^2}{(x+1)\sqrt{x+1}}\,dx=\int \frac{(t^2-1)^2}{t^2\cdot t}\,2t\,dt$$

$$=2\int \frac{(t^2-1)^2}{t^2}\,dt$$

$$=2\int \left(t^2-2+\frac{1}{t^2}\right)dt$$

$$=2\left(\frac{1}{3}t^3-2t-\frac{1}{t}\right)+C$$

$$=\frac{2}{3t}(t^4-6t^2-3)+C$$

$$=\frac{2(x^2-4x-8)}{3\sqrt{x+1}}+C$$

O 84 b

(3) $\int (3+2x)\sqrt{1+2x}\,dx$

[Sol] Letting $t=\sqrt{1+2x}$, $x=\frac{t^2-1}{2}$

$$\frac{dx}{dt}=t,\text{ therefore, } dx=tdt$$

$$\begin{aligned}\therefore \int (3+2x)\sqrt{1+2x}\,dx &= \int (t^2+2)t\cdot t\,dt\\ &= \int (t^4+2t^2)\,dt\\ &= \frac{1}{5}t^5+\frac{2}{3}t^3+C\\ &= \frac{1}{5}(1+2x)^2\sqrt{1+2x}+\frac{2}{3}(1+2x)\sqrt{1+2x}+C\\ &= \frac{1}{15}(1+2x)(13+6x)\sqrt{1+2x}+C\end{aligned}$$

(4) $\int (x^3+x)\sqrt{1+x^2}\,dx$

[Sol] Letting $t=1+x^2$,

$$\frac{dt}{dx}=2x,\text{ therefore, } 2xdx=dt$$

$$\begin{aligned}\therefore \int (x^3+x)\sqrt{1+x^2}\,dx &= \int x(x^2+1)\sqrt{1+x^2}\,dx\\ &= \int t\sqrt{t}\cdot\frac{1}{2}dt\\ &= \frac{1}{2}\int t^{\frac{3}{2}}dt\\ &= \frac{1}{5}t^{\frac{5}{2}}+C\\ &= \frac{1}{5}(1+x^2)^2\sqrt{1+x^2}+C\end{aligned}$$

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

Evaluate each of the following indefinite integrals by applying the method of substitution.

(1) $\displaystyle\int \frac{e^x}{e^x+1}dx$

[Sol] Letting $t = e^x$, $\boxed{e^x}\,dx = dt$

$$\therefore \int \frac{e^x}{e^x+1}dx = \int \boxed{\frac{1}{t+1}}\,dt$$

$$= \ln|t+1| + C$$

$$= \ln(e^x+1) + C$$

(2) $\displaystyle\int \frac{dx}{e^x - e^{-x}}$

[Sol] Letting $t = e^x$, $e^x\,dx = dt$

$$\therefore \int \frac{dx}{e^x - e^{-x}} = \int \frac{e^x}{e^{2x}-1}dx$$

$$= \int \frac{dt}{t^2-1}$$

$$= \frac{1}{2}\int\left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt$$

$$= \frac{1}{2}\{\ln|t-1| - \ln|t+1|\} + C$$

$$= \frac{1}{2}\{\ln|e^x-1| - \ln|e^x+1|\} + C$$

$$\left[= \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C\right]$$

O 85 b

(3) $\displaystyle\int \frac{\ln x}{x} dx$

[Sol] Letting $t = \ln x$, $\frac{1}{x}dx = dt$

$$\therefore \int \frac{\ln x}{x} dx = \int t\,dt$$

$$= \frac{1}{2}t^2 + C$$

$$= \frac{1}{2}(\ln x)^2 + C$$

(4) $\displaystyle\int \frac{dx}{x \ln x}$

[Sol] Letting $t = \ln x$, $\frac{1}{x}dx = dt$

$$\therefore \int \frac{dx}{x \ln x} = \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$$

$$= \int \frac{1}{t} dt$$

$$= \ln|t| + C$$

$$= \ln|\ln x| + C$$

(5) $\displaystyle\int \frac{\ln x}{x(1 + \ln x)} dx$

[Sol] Letting $t = \ln x$, $\frac{1}{x}dx = dt$

$$\therefore \int \frac{\ln x}{x(1 + \ln x)} dx = \int \frac{t}{1+t} dt$$

$$= \int \left(1 - \frac{1}{t+1}\right) dt$$

$$= t - \ln|t+1| + C$$

$$= \ln x - \ln|\ln x + 1| + C$$

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O 86 a

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Evaluate each of the following indefinite integrals by applying the method of substitution.

Ex. $\int \sin^2 x \cos x\,dx$

[Sol] Letting $t = \sin x$, …①

$$\frac{dt}{dx} = \boxed{\cos x}$$

$$\boxed{\cos x\,dx} = dt \quad \text{…②}$$

$$\therefore \int \sin^2 x \cos x\,dx = \int \boxed{t^2\,dt}$$

Use ① and ② to express the given integral in terms of t.

$$= \boxed{\frac{1}{3}t^3 + C}$$

$$= \boxed{\frac{1}{3}\sin^3 x + C}$$

Express this value in terms of the original variable, x.

Answers: $\cos x$, $\cos x\,dx$, $t^2\,dt$, $\frac{1}{3}t^3 + C$, $\frac{1}{3}\sin^3 x + C$

(1) $\int \cos^3 x \sin x\,dx$

[Sol] Letting $t = \cos x$, $-\sin x\,dx = dt$

$$\therefore \int \cos^3 x \sin x\,dx = -\int t^3\,dt = -\frac{1}{4}t^4 + C = -\frac{1}{4}\cos^4 x + C$$

(2) $\int (1 + \cos x + \cos^2 x)^2 \sin x\,dx$

[Sol] Letting $t = \cos x$, $-\sin x\,dx = dt$

$$\therefore \int (1 + \cos x + \cos^2 x)^2 \sin x\,dx = -\int (1 + t + t^2)^2\,dt$$

$$= -\int (1 + 2t + 3t^2 + 2t^3 + t^4)\,dt$$

$$= -\left(t + t^2 + t^3 + \frac{1}{2}t^4 + \frac{1}{5}t^5\right) + C$$

$$= -\frac{1}{5}\cos^5 x - \frac{1}{2}\cos^4 x - \cos^3 x - \cos^2 x - \cos x + C$$

O 86 b

(3) $\int \sin^3 x dx$

[Sol] $\int \sin^3 x dx = \int \boxed{\sin^2 x} \cdot \sin x dx$

$$= \int (1 - \boxed{\cos^2 x})\sin x dx$$

Letting $t = \cos x$, $\boxed{-\sin x} dx = dt$

$$\therefore \int \sin^3 x dx = \int \boxed{(1-t^2)} (-1) dt$$

$$= \int (t^2 - 1) dt$$

$$= \frac{t^3}{3} - t + C$$

$$= \frac{1}{3}\cos^3 x - \cos x + C$$

(4) $\int \sin^5 x dx$

[Sol] $\int \sin^5 x dx = \int \sin^4 x \cdot \sin x dx$

$$= \int (1 - \cos^2 x)^2 \sin x dx$$

Letting $t = \cos x$, $-\sin x dx = dt$

$$\therefore \int \sin^5 x dx = \int (1-t^2)^2 (-1) dt$$

$$= -\int (1 - 2t^2 + t^4) dt$$

$$= -t + \frac{2}{3}t^3 - \frac{1}{5}t^5 + C$$

$$= -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$$

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O 87 a

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	1	2~

Evaluate each of the following indefinite integrals by applying the method of substitution.

(1) $\displaystyle\int \tan x\,dx$ (**Hint:** Let $t = \cos x$)

[Sol] Letting $t = \cos x$, $-\sin x\,dx = dt$

$$\therefore \int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$$
$$= \int \frac{1}{t}(-1)dt$$
$$= -\int \frac{dt}{t}$$
$$= -\ln|t| + C$$
$$= -\ln|\cos x| + C$$

(2) $\displaystyle\int \frac{\cos^3 x}{\sin^2 x}dx$

[Sol] Letting $t = \sin x$, $\cos x\,dx = dt$

$$\therefore \int \frac{\cos^3 x}{\sin^2 x}dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^2 x}dx$$
$$= \int \frac{1-t^2}{t^2}dt$$
$$= \int \left(\frac{1}{t^2} - 1\right)dt$$
$$= -\frac{1}{t} - t + C$$
$$= -\frac{1}{\sin x} - \sin x + C$$

O 87 b

(3) $\displaystyle\int \tan^3 x\,dx$

[Sol] Letting $t = \cos x$, $\quad -\sin x\,dx = dt$

$$\therefore \int \tan^3 x\,dx = \int \frac{\sin^3 x}{\cos^3 x}\,dx = \int \frac{(1-\cos^2 x)\sin x}{\cos^3 x}\,dx$$

$$= \int \frac{t^2-1}{t^3}\,dt = \int \left(\frac{1}{t} - \frac{1}{t^3}\right)dt$$

$$= \ln|t| + \frac{1}{2t^2} + C = \ln|\cos x| + \frac{1}{2\cos^2 x} + C$$

Alternate Solution

[Sol] $\displaystyle\int \tan^3 x\,dx = \int (\tan^2 x)\cdot\tan x\,dx = \int\left(\frac{1}{\cos^2 x} - 1\right)\cdot\tan x\,dx$

$$= \int \frac{\tan x}{\cos^2 x}\,dx - \int \tan x\,dx$$

Letting $t = \tan x$, $\quad \dfrac{1}{\cos^2 x}\,dx = dt$

$$\therefore \int \tan^3 x\,dx = \int \frac{\tan x}{\cos^2 x}\,dx - \int \tan x\,dx = \int t\,dt - \int \tan x\,dx$$

$$= \frac{t^2}{2} - (-\ln|\cos x|) + C = \frac{\tan^2 x}{2} + \ln|\cos x| + C$$

(4) $\displaystyle\int (2+\cos x)\sin^3 x\,dx$

[Sol] Letting $t = \cos x$, $\quad -\sin x\,dx = dt$

$$\therefore \int (2+\cos x)\sin^3 x\,dx = \int (2+\cos x)(1-\cos^2 x)\sin x\,dx$$

$$= \int (2+t)(1-t^2)(-1)\,dt$$

$$= \int (t^3 + 2t^2 - t - 2)\,dt$$

$$= \frac{1}{4}t^4 + \frac{2}{3}t^3 - \frac{1}{2}t^2 - 2t + C$$

$$= \frac{1}{4}\cos^4 x + \frac{2}{3}\cos^3 x - \frac{1}{2}\cos^2 x - 2\cos x + C$$

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	–	1	2–

Evaluate each of the following indefinite integrals by applying the method of substitution.

(1) $\displaystyle\int \frac{\sin^3 x}{\cos^5 x}dx$

[Sol] Letting $t=\cos x$, $-\sin x\,dx = dt$

$$\therefore \int \frac{\sin^3 x}{\cos^5 x}dx = \int \frac{(1-\cos^2 x)\sin x}{\cos^5 x}dx$$
$$= \int \frac{1-t^2}{t^5}(-1)dt$$
$$= \int \frac{t^2-1}{t^5}dt$$
$$= \int\left(\frac{1}{t^3}-\frac{1}{t^5}\right)dt$$
$$= -\frac{1}{2t^2}+\frac{1}{4t^4}+C$$
$$= \frac{1}{4\cos^4 x}-\frac{1}{2\cos^2 x}+C$$

Alternate Solution

$$\int \frac{\sin^3 x}{\cos^5 x}dx = \int \tan^3 x\cdot\frac{1}{\cos^2 x}dx$$
$$= \int \tan^3 x\cdot(\tan x)'dx$$
$$= \frac{1}{4}\tan^4 x + C$$

(2) $\displaystyle\int \cos^3 x\sin^2 x\,dx$

[Sol] Letting $t=\sin x$, $\cos x\,dx = dt$

$$\therefore \int \cos^3 x\sin^2 x\,dx = \int \sin^2 x(1-\sin^2 x)\cos x\,dx$$
$$= \int t^2(1-t^2)dt$$
$$= \int (t^2-t^4)dt$$
$$= \frac{1}{3}t^3-\frac{1}{5}t^5+C$$
$$= \frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$$

O 88 b

(3) $\int \frac{dx}{\sin x}$

[Sol] $\int \frac{dx}{\sin x} = \int \frac{\sin x}{\boxed{\sin^2 x}} dx$

$= \int \frac{\sin x}{1 - \boxed{\cos^2 x}} dx$

Letting $t = \cos x$, $\boxed{-\sin x} dx = dt$

$\therefore \int \frac{dx}{\sin x} = \int \boxed{\frac{1}{1-t^2}} (-1) dt$

$= \int \frac{dt}{t^2 - 1}$

$= \frac{1}{2} \int \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt$

$= \frac{1}{2} (\ln|t-1| - \ln|t+1|) + C$

$= \frac{1}{2} (\ln|\cos x - 1| - \ln|\cos x + 1|) + C \quad \left[= \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C \right]$

(4) $\int \frac{dx}{\cos x}$

[Sol] $\int \frac{dx}{\cos x} = \int \frac{\cos x}{\cos^2 x} dx$

$= \int \frac{\cos x}{1 - \sin^2 x} dx$

Letting $t = \sin x$, $\cos x\, dx = dt$

$\therefore \int \frac{dx}{\cos x} = \int \frac{dt}{1 - t^2}$

$= -\int \frac{dt}{t^2 - 1}$

$= \frac{1}{2} \int \left(\frac{1}{t+1} - \frac{1}{t-1} \right) dt$

$= \frac{1}{2} (\ln|t+1| - \ln|t-1|) + C$

$= \frac{1}{2} (\ln|\sin x + 1| - \ln|\sin x - 1|) + C \quad \left[= \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C \right]$

O 89 a

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69% -
(mistakes) 0	-	-	1	2-

Evaluate each of the following indefinite integrals.

(1) $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$ (**Hint:** Let $t = 1 + \sin x$)

[Sol] Letting $t = 1 + \sin x$, $\cos x\,dx = dt$

$$\therefore \int \frac{\cos x}{1+\sin x}\,dx = \int \frac{dt}{t}$$
$$= \ln|t| + C$$
$$= \ln(1+\sin x) + C$$

(2) $\displaystyle\int \frac{1+\cos x}{x+\sin x}\,dx$

[Sol] Letting $t = x + \sin x$, $(1+\cos x)\,dx = dt$

$$\therefore \int \frac{1+\cos x}{x+\sin x}\,dx = \int \frac{dt}{t}$$
$$= \ln|t| + C$$
$$= \ln|x+\sin x| + C$$

O 89 b

(3) $\displaystyle\int \frac{dx}{1-\sin x}$

[Sol] $\displaystyle\int \frac{dx}{1-\sin x} = \int \frac{\boxed{1+\sin x}}{(1-\sin x)(1+\sin x)}dx$

$$= \int \frac{\boxed{1+\sin x}}{1-\sin^2 x}dx$$

$$= \int \left(\frac{\boxed{1}}{\cos^2 x} + \frac{\boxed{\sin x}}{\cos^2 x}\right)dx$$

Since $\displaystyle\int \frac{dx}{\cos^2 x} = \boxed{\tan x} + C_1,$ and

$$\int \frac{\boxed{\sin x}}{\cos^2 x}dx = \int \frac{(-\cos x)'}{\cos^2 x}dx = \boxed{\frac{1}{\cos x}} + C_2$$

Letting $C = C_1 + C_2$,

$$\int \frac{dx}{1-\sin x} = \boxed{\tan x} + \boxed{\frac{1}{\cos x}} + \boxed{C}$$

(4) $\displaystyle\int \frac{\cos x}{1+\cos x}dx$

[Sol] $\displaystyle\int \frac{\cos x}{1+\cos x}dx = \int \frac{\cos x(1-\cos x)}{(1+\cos x)(1-\cos x)}dx$

$$= \int \frac{\cos x - \cos^2 x}{1-\cos^2 x}dx$$

$$= \int \frac{\cos x - (1-\sin^2 x)}{\sin^2 x}dx$$

$$= \int \left(\frac{\cos x}{\sin^2 x} - \frac{1}{\sin^2 x} + 1\right)dx$$

$$= -\frac{1}{\sin x} + \frac{1}{\tan x} + x + C$$

KUMON

O 90 a

Indefinite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2–

Evaluate each of the following indefinite integrals.

(1) $\int (3x-2)^5 dx$

[Sol] Letting $t = 3x - 2$, $x = \frac{t+2}{3}$

$$\frac{dx}{dt} = \frac{1}{3}, \text{ therefore, } dx = \frac{1}{3}dt$$

$$\therefore \int (3x-2)^5 dx = \int t^5 \cdot \frac{1}{3} dt$$

$$= \frac{1}{18}t^6 + C$$

$$= \frac{1}{18}(3x-2)^6 + C$$

(2) $\int \frac{x+1}{x\sqrt{2x+1}} dx$

[Sol] Letting $t = \sqrt{2x+1}$, $x = \frac{t^2-1}{2}$ and $dx = t\,dt$

$$\therefore \int \frac{x+1}{x\sqrt{2x+1}} dx = \int \frac{\frac{t^2-1}{2}+1}{\frac{t^2-1}{2} \cdot t} t\,dt$$

$$= \int \frac{t^2+1}{t^2-1} dt$$

$$= \int \left(1 + \frac{2}{t^2-1}\right) dt$$

$$= \int \left(1 + \frac{1}{t-1} - \frac{1}{t+1}\right) dt$$

$$= t + \ln|t-1| - \ln|t+1| + C$$

$$= \sqrt{2x+1} + \ln\left|\sqrt{2x+1}-1\right| - \ln\left(\sqrt{2x+1}+1\right) + C$$

(Alternate answer: $= \sqrt{2x+1} + \ln\left|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right| + C$

$= \sqrt{2x+1} + \ln\left|\frac{x+1-\sqrt{2x+1}}{x}\right| + C$)

90 b

(3) $\displaystyle\int \frac{\sin x}{\sqrt{\cos x}}\,dx$

[Sol] Letting $t = \cos x$, $\quad -\sin x\,dx = dt$

$$\therefore \int \frac{\sin x}{\sqrt{\cos x}}\,dx = \int \frac{1}{\sqrt{t}}(-1)\,dt$$
$$= -\int t^{-\frac{1}{2}}\,dt$$
$$= -2t^{\frac{1}{2}} + C$$
$$= -2\sqrt{\cos x} + C$$

(4) $\displaystyle\int \frac{\sin x}{1-\sin x}\,dx$

[Sol] $$\int \frac{\sin x}{1-\sin x}\,dx = \int \frac{\sin x(1+\sin x)}{(1-\sin x)(1+\sin x)}\,dx$$
$$= \int \frac{\sin x + \sin^2 x}{1-\sin^2 x}\,dx$$
$$= \int \frac{\sin x + 1 - \cos^2 x}{\cos^2 x}\,dx$$
$$= \int \left(\frac{\sin x}{\cos^2 x} + \frac{1}{\cos^2 x} - 1\right)dx$$
$$= \frac{1}{\cos x} + \tan x - x + C$$

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

From Level L, since $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$,

Integrating the RHS, $\int f'(x)g(x)dx + \int f(x)g'(x)dx = f(x)g(x)$

Therefore, $\boxed{\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx}$

This rule of integration is called *Integration by Parts.*

Evaluate each of the following indefinite integrals by using the rule of Integration by Parts.

Ex. $\int x\sin x dx$

[Sol] Letting $f'(x) = \sin x$, and $g(x) = x$,
$f(x) = -\cos x$

Note: Since $g(x) = x$,
$g'(x) = 1$
Therefore, $g'(x) = \underline{x' = 1}$

Therefore,

$$\int x\sin x dx = \int(\sin x)x dx = \int(-\cos x)' x dx$$

$$= (-\cos x)x - \int(-\cos x)x' dx$$

$$= -x\cos x + \sin x + C$$

(1) $\int x\cos x dx$

[Sol] $\int x\cos x dx = \int x(\sin x)' dx$

$$= x\sin x - \int \sin x dx$$

$$= x\sin x + \cos x + C$$

O 91 b

(2) $$\int x\cos 3x\,dx = \int x\left(\frac{\sin 3x}{3}\right)' dx$$

$$= \frac{1}{3}x\sin 3x - \int \frac{\sin 3x}{3}dx$$

$$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$$

(3) $$\int xe^x\,dx = \int x(e^x)'\,dx$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

$$[= e^x(x-1) + C]$$

(4) $$\int xe^{2x}dx = \int x\left(\frac{e^{2x}}{2}\right)' dx$$

$$= \frac{1}{2}xe^{2x} - \int \frac{e^{2x}}{2}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$\left[= \frac{1}{2}e^{2x}\left(x - \frac{1}{2}\right) + C\right]$$

(5) $$\int x(x-1)^5\,dx = \int x\left[\frac{(x-1)^6}{6}\right]' dx$$

$$= \frac{1}{6}x(x-1)^6 - \frac{1}{6}\int (x-1)^6 dx$$

$$= \frac{1}{6}x(x-1)^6 - \frac{1}{42}(x-1)^7 + C$$

$$= \frac{1}{42}(6x+1)(x-1)^6 + C$$

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

Evaluate each of the following indefinite integrals by using the rule of Integration by Parts.

(1) $\int x \ln x \, dx = \int \left(\boxed{\frac{x^2}{2}} \right)' \ln x \, dx$

$= \boxed{\frac{1}{2}x^2} \ln x - \int \boxed{\frac{1}{2}x} \, dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

(2) $\int \ln x \, dx = \int (\boxed{x})' \ln x \, dx$

$= \boxed{x} \ln x - \int \boxed{dx}$

$= x \ln x - x + C$

(3) $\int \ln(2x+1) dx = \int (x)' \ln(2x+1) dx$

$= x \ln(2x+1) - \int \frac{2x}{2x+1} dx$

$= x \ln(2x+1) - \int \left(1 - \frac{1}{2x+1}\right) dx$

$= x \ln(2x+1) - x + \frac{1}{2} \ln(2x+1) + C$

$= \frac{1}{2}(2x+1) \ln(2x+1) - x + C$

O 92 b

(4) $\int x^2 \sin x dx = \int x^2(-\cos x)' dx$

$$= x^2(-\cos x) - \int 2x(-\cos x) dx$$

$$= -x^2\cos x + 2\int x(\sin x)' dx$$

$$= -x^2\cos x + 2\left(x\sin x - \int \sin x dx\right)$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

(5) $\int x^2 \cos x dx = \int x^2(\sin x)' dx$

$$= x^2\sin x - \int 2x\sin x dx$$

$$= x^2\sin x + 2\int x(\cos x)' dx$$

$$= x^2\sin x + 2\left(x\cos x - \int \cos x dx\right)$$

$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

(6) $\int \frac{x}{\cos^2 x} dx = \int x(\tan x)' dx$

$$= x\tan x - \int \tan x dx$$

$$= x\tan x - \int \frac{(-\cos x)'}{\cos x} dx$$

$$= x\tan x + \ln|\cos x| + C$$

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

Evaluate each of the following indefinite integrals by using the rule of Integration by Parts.

(1) $\int x^4 \ln x dx = \int \left(\frac{x^5}{5}\right)' \ln x dx$

$= \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^4 dx$

$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C$

(2) $\int \log_a x dx = \int \frac{\boxed{\ln x}}{\ln a} dx$

$= \frac{\boxed{1}}{\ln a}\int \boxed{(x)'\ln x}\, dx$

$= \frac{1}{\ln a}\left(x \ln x - \int dx\right)$

$= \frac{1}{\ln a}(x \ln x - x) + C$

$\left[= x \log_a \frac{x}{e} + C\right]$

(3) $\int (\ln x)^2 dx = \int (x)'(\ln x)^2 dx$

$= x(\ln x)^2 - 2\int x \ln x \cdot \frac{1}{x} dx$

$= x(\ln x)^2 - 2\int \ln x dx$

$= x(\ln x)^2 - 2\int (x)' \ln x dx$

$= x(\ln x)^2 - 2\left(x \ln x - \int dx\right)$

$= x(\ln x)^2 - 2x \ln x + 2x + C$

O 93 b

(4) $\int \sqrt{x}\ln x\,dx = \int\left(\frac{2}{3}x^{\frac{3}{2}}\right)'\ln x\,dx$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \int\frac{2}{3}x^{\frac{3}{2}}\cdot\frac{1}{x}dx$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}dx$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\cdot\frac{2}{3}x^{\frac{3}{2}} + C$

$= \frac{2}{3}x\sqrt{x}\ln x - \frac{4}{9}x\sqrt{x} + C$

(5) $\int x^2e^x dx = \int x^2(e^x)'dx$

$= x^2e^x - \int 2xe^x dx$

$= x^2e^x - 2\int x(e^x)'dx$

$= x^2e^x - 2\left(xe^x - \int e^x dx\right)$

$= x^2e^x - 2xe^x + 2e^x + C$

$[= e^x(x^2 - 2x + 2) + C]$

(6) $\int x^3e^x dx = \int x^3(e^x)'dx$

$= x^3e^x - \int 3x^2e^x dx$

$= x^3e^x - 3\int x^2(e^x)'dx$

$= x^3e^x - 3\left(x^2e^x - \int 2xe^x dx\right)$

$= x^3e^x - 3x^2e^x + 6\int x(e^x)'dx$

$= x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x dx\right)$

$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

$[= e^x(x^3 - 3x^2 + 6x - 6) + C]$

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	–	1	2-

Evaluate each of the following indefinite integrals by using the rule of Integration by Parts.

Ex.

$$\int e^x \sin x\,dx = \int (e^x)' \sin x\,dx$$
$$= e^x \sin x - \int e^x \cos x\,dx$$
$$= e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right)$$
$$= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

Therefore, $2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$

And, thus, $\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$

(1) $\int e^x \cos x\,dx = \int (e^x)' \cos x\,dx$

$$= e^x \cos x + \int e^x \sin x\,dx$$
$$= e^x \cos x + \left(e^x \sin x - \int e^x \cos x\,dx\right)$$

Therefore, $2\int e^x \cos x\,dx = e^x(\sin x + \cos x)$

And, thus, $\int e^x \cos x\,dx = \frac{e^x}{2}(\sin x + \cos x) + C$

(2) $\int \frac{\ln x}{x}\,dx = \int (\ln x)' \ln x\,dx$

$$= (\ln x)^2 - \int \frac{\ln x}{x}\,dx$$

Therefore, $2\int \frac{\ln x}{x}\,dx = (\ln x)^2$

And, thus, $\int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + C$

O 94 b

(3) $\int x^2(\ln x)^2 dx = \int \left(\frac{x^3}{3}\right)' (\ln x)^2 dx$

$$= \frac{1}{3}x^3(\ln x)^2 - \int \frac{1}{3}x^3 \cdot (2\ln x) \cdot \frac{1}{x} dx$$

$$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int x^2 \ln x dx$$

$$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int \left(\frac{x^3}{3}\right)' \ln x dx$$

$$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx\right)$$

$$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{9}\int x^2 dx$$

$$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C$$

$$\left[= \frac{1}{3}x^3\left[(\ln x)^2 - \frac{2}{3}\ln x + \frac{2}{9}\right] + C \right]$$

(4) $\int e^{2x} \sin 3x dx = \int \left(\frac{e^{2x}}{2}\right)' \sin 3x dx$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x dx$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\left(\frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x dx\right)$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x dx$$

Therefore, $\frac{13}{4}\int e^{2x}\sin 3x dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x$

And, thus, $\int e^{2x}\sin 3x dx = \frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$

KUMON

O 95 a

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Evaluate each of the following indefinite integrals by using the rule of Integration by Parts.

(1) $\int x\ln(x^2+1)dx = \int\left(\boxed{\dfrac{x^2+1}{2}}\right)'\ln(\boxed{x^2+1})dx$

$$= \frac{1}{2}(x^2+1)\ln(x^2+1) - \int\frac{x^2+1}{2}\cdot\frac{2x}{x^2+1}dx$$

$$= \frac{1}{2}(x^2+1)\ln(x^2+1) - \int x\,dx$$

$$= \frac{1}{2}(x^2+1)\ln(x^2+1) - \frac{1}{2}x^2 + C$$

(2) $\int e^{2x}\cos 3x\,dx = \int\left(\dfrac{e^{2x}}{2}\right)'\cos 3x\,dx$

$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx$$

$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right)$$

$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x\,dx$$

Therefore, $\dfrac{13}{4}\int e^{2x}\cos 3x\,dx = \dfrac{1}{2}e^{2x}\cos 3x + \dfrac{3}{4}e^{2x}\sin 3x$

And, thus, $\int e^{2x}\cos 3x\,dx = \dfrac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x) + C$

O 95 b

(3) $\int x^2 e^{1-x}dx = \int x^2(-e^{1-x})'dx$

$$= -x^2e^{1-x} + \int 2xe^{1-x}dx$$

$$= -x^2e^{1-x} + 2\left(-xe^{1-x} + \int e^{1-x}dx\right)$$

$$= -x^2e^{1-x} - 2xe^{1-x} - 2e^{1-x} + C$$

$$[= -e^{1-x}(x^2 + 2x + 2) + C]$$

(4) $\int (x^2 - x)e^{-x}dx = \int (x^2 - x)(-e^{-x})'dx$

$$= -(x^2 - x)e^{-x} + \int (2x - 1)(e^{-x})dx$$

$$= -(x^2 - x)e^{-x} - (2x - 1)e^{-x} + 2\int e^{-x}dx$$

$$= -(x^2 - x)e^{-x} - (2x - 1)e^{-x} - 2e^{-x} + C$$

$$= -e^{-x}[(x^2 - x) + (2x - 1) + 2] + C$$

$$[= -e^{-x}(x^2 + x + 1) + C]$$

O 96 a

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	—	—	—	—

1. Evaluate the following indefinite integral as shown in the example.

Ex. $\displaystyle\int\frac{dx}{x^2-4}$

[Sol] $\displaystyle\frac{1}{x^2-4}=\frac{1}{(x+2)(x-2)}$ —①

Expressing the RHS of ① as a sum of fractions,

$$\frac{1}{(x+2)(x-2)}=\frac{P}{x+2}+\frac{Q}{x-2} \quad —②$$

$$P(x-2)+Q(x+2)=1 \quad —③$$

From ③, When $x=2$, $4Q=1$, $Q=\dfrac{1}{4}$

When $x=-2$, $-4P=1$, $P=-\dfrac{1}{4}$

Substituting P and Q into ②,

$$\frac{1}{(x+2)(x-2)}=\frac{1}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right)$$

$$\therefore \int\frac{dx}{x^2-4}=\frac{1}{4}(\ln|x-2|-\ln|x+2|)+C$$

$$\left[=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C\right]$$

(1) $\displaystyle\int\frac{dx}{x^2-9}$

[Sol] $\displaystyle\frac{1}{x^2-9}=\frac{1}{(x+3)(x-3)}$ —①

Expressing the RHS of ① as a sum of fractions,

$$\frac{1}{(x+3)(x-3)}=\frac{P}{x+3}+\frac{Q}{x-3} \quad —②$$

$$P(x-3)+Q(x+3)=1 \quad —③$$

From ③, When $x=3$, $6Q=1$, $Q=\dfrac{1}{6}$

When $x=-3$, $-6P=1$, $P=-\dfrac{1}{6}$

Substituting P and Q into ②,

$$\frac{1}{(x+3)(x-3)}=\frac{1}{6}\left(\frac{1}{x-3}-\frac{1}{x+3}\right)$$

$$\therefore \int\frac{dx}{x^2-9}=\frac{1}{6}(\ln|x-3|-\ln|x+3|)+C \qquad \left[=\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|+C\right]$$

O 96 b

2. Evaluate each of the following indefinite integrals.

(1) $\displaystyle\int \frac{x+3}{2x^2-3x-2}dx$

[Sol] $\displaystyle\frac{x+3}{2x^2-3x-2}=\frac{x+3}{(x-2)(\boxed{2x+1})}$

Letting $\displaystyle\frac{x+3}{(x-2)(\boxed{2x+1})}=\frac{P}{x-2}+\frac{Q}{\boxed{2x+1}}$.

Determining the values of P and Q.

$x+3=P(2x+1)+Q(x-2)$

$x+3=(2P+Q)x+P-2Q$ …①

Matching the coefficients of the left- and right-hand sides of equation ①.

$\begin{cases} 2P+Q=1 \\ P-2Q=3 \end{cases}$ $\quad\therefore P=1,\ \ Q=-1$

$$\therefore \int \frac{x+3}{2x^2-3x-2}dx=\int\left(\frac{1}{x-2}-\frac{1}{2x+1}\right)dx$$

$$=\ln|x-2|-\frac{1}{2}\ln|2x+1|+C$$

$$\left[=\frac{1}{2}\ln\frac{(x-2)^2}{|2x+1|}+C\right]$$

(2) $\displaystyle\int \frac{2x}{x^2+3x+2}dx$

[Sol] $\displaystyle\frac{2x}{x^2+3x+2}=\frac{2x}{(x+1)(x+2)}$

Letting $\displaystyle\frac{2x}{(x+1)(x+2)}=\frac{P}{x+1}+\frac{Q}{x+2}$.

Determining the values of P and Q.

$2x=P(x+2)+Q(x+1)$

$2x=(P+Q)x+2P+Q$ …①

Matching the coefficients of the left- and right-hand sides of equation ①.

$\begin{cases} P+Q=2 \\ 2P+Q=0 \end{cases}$ $\quad\therefore P=-2,\ \ Q=4$

$$\therefore \int \frac{2x}{x^2+3x+2}dx=\int\left(-\frac{2}{x+1}+\frac{4}{x+2}\right)dx$$

$$=-2(\ln|x+1|-2\ln|x+2|)+C$$

$$\left[=-2\ln\frac{|x+1|}{(x+2)^2}+C=2\ln\frac{(x+2)^2}{|x+1|}+C\right]$$

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

1. Evaluate each of the following indefinite integrals.

(1) $\displaystyle\int \frac{x^4}{x^2-1}dx = \int\left(x^2+1+\frac{1}{\boxed{x^2-1}}\right)dx$

$$= \int\left[x^2+1+\frac{1}{2}\left(\frac{1}{\boxed{x-1}}-\frac{1}{\boxed{x+1}}\right)\right]dx$$

$$= \frac{1}{3}x^3+x+\frac{1}{2}(\ln|x-1|-\ln|x+1|)+C$$

$$\left[=\frac{1}{3}x^3+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C\right]$$

(2) $\displaystyle\int \frac{x^2+1}{x^2+x}dx$

[Sol] $\dfrac{x^2+1}{x^2+x} = 1-\dfrac{x-1}{x^2+x}$

Letting $\dfrac{x-1}{x^2+x} = \dfrac{P}{x}+\dfrac{Q}{x+1}$,

$x-1 = P(x+1)+Qx$

$x-1 = (P+Q)x+P$ …①

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P+Q=1 \\ P=-1 \end{cases} \quad \therefore P=-1,\ Q=2$$

$$\therefore \int\frac{x^2+1}{x^2+x}dx = \int\left(1+\frac{1}{x}-\frac{2}{x+1}\right)dx$$

$$= x+\ln|x|-2\ln|x+1|+C$$

$$\left[= x+\ln\frac{|x|}{(x+1)^2}+C\right]$$

O 97 b

2. Given that $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$, evaluate each of the following indefinite integrals.

(1) $\int \frac{3x^2}{x^3}\,dx = \int \frac{(x^3)'}{x^3}\,dx$

$= \ln|x^3| + C$

(2) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$= \int \frac{-(\cos x)'}{\cos x}\,dx$

$= -\ln|\cos x| + C$

(3) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{(e^x + e^{-x})'}{e^x + e^{-x}}\,dx$

$= \ln(e^x + e^{-x}) + C$

(4) $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = \int \frac{-(\sin x + \cos x)'}{\sin x + \cos x}\,dx$

$= -\ln|\sin x + \cos x| + C$

KUMON

O 98 a

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	1	2-

Evaluate each of the following indefinite integrals by using the method of substitution.

(1) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

[Sol] Letting $t = \sqrt{x+1}$, $x = t^2 - 1$ and $dx = 2t\,dt$

$$\therefore \int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{t^2-1}{t} 2t \cdot dt$$
$$= 2\int (t^2-1)\,dt$$
$$= \frac{2}{3}t^3 - 2t + C$$
$$= \frac{2}{3}(x+1)\sqrt{x+1} - 2\sqrt{x+1} + C$$
$$= \frac{2}{3}(x-2)\sqrt{x+1} + C$$

(2) $\displaystyle\int \frac{x}{\sqrt{x^2-1}}\,dx$

[Sol] Letting $t = \sqrt{x^2-1}$, $x^2 - 1 = t^2$

Therefore, $2x\,dx = 2t\,dt$
And, thus, $x\,dx = t\,dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}\,dx = \int \frac{1}{t} \cdot t\,dt$$
$$= \int dt$$
$$= t + C$$
$$= \sqrt{x^2-1} + C$$

O 98 b

(3) $\displaystyle\int \frac{x}{1+x^2}\ln(1+x^2)\,dx$

[Sol] Letting $u = 1 + x^2$, $\quad du = 2x\,dx$

$$\int \frac{x}{1+x^2}\ln(1+x^2)\,dx = \frac{1}{2}\int \frac{\ln u}{u}\,du$$

Letting $t = \ln u$, $\quad dt = \frac{1}{u}\,du$

$$\begin{aligned}\frac{1}{2}\int \frac{\ln u}{u}\,du &= \frac{1}{2}\int t\,dt \\ &= \frac{1}{2}\left[\frac{t^2}{2}\right] + C \\ &= \frac{t^2}{4} + C \\ &= \frac{(\ln u)^2}{4} + C \\ &= \frac{[\ln(1+x^2)]^2}{4} + C\end{aligned}$$

(4) $\displaystyle\int \frac{x}{1+x^2}\ln\left(\sqrt{1+x^2}\right)dx$

[Sol] Letting $u = \sqrt{1+x^2}$, $\quad du = \frac{x}{\sqrt{1+x^2}}\,dx$

$$\begin{aligned}\int \frac{x}{1+x^2}\ln\left(\sqrt{1+x^2}\right)dx &= \int \frac{1}{\sqrt{1+x^2}}\cdot\frac{x}{\sqrt{1+x^2}}\ln\left(\sqrt{1+x^2}\right)dx \\ &= \int \frac{\ln u}{u}\,du\end{aligned}$$

Letting $t = \ln u$, $\quad dt = \frac{1}{u}\,du$

$$\begin{aligned}\int \frac{\ln u}{u}\,du &= \int t\,dt \\ &= \frac{t^2}{2} + C \\ &= \frac{(\ln u)^2}{2} + C \\ &= \frac{\left[\ln\left(\sqrt{1+x^2}\right)\right]^2}{2} + C\end{aligned}$$

KUMON

O 99 a

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	–	–	–

Evaluate each of the following indefinite integrals by using the method of substitution.

(1) $\displaystyle\int\frac{dx}{\sqrt{x^2+a^2}}$

[Sol] Letting $t=x+\sqrt{x^2+a^2}$ (where $t>0$),

$$x^2+a^2=(t-x)^2$$
$$2xt=t^2-a^2$$
$$\therefore\ x=\frac{t^2-a^2}{2t}$$

Therefore, $\sqrt{x^2+a^2}=t-x=\dfrac{t^2+a^2}{2t}$

$$\frac{dx}{dt}=\frac{t^2+a^2}{2t^2}$$

Thus, $\displaystyle\int\frac{dx}{\sqrt{x^2+a^2}}=\int\frac{2t}{t^2+a^2}\cdot\frac{t^2+a^2}{2t^2}\,dt$

$$=\int\frac{dt}{t}$$
$$=\ln|t|+C$$
$$=\ln\left|x+\sqrt{x^2+a^2}\right|+C$$

O 99 b

O

(2) $\int \frac{dx}{\sqrt{x^2+1}}$

(3

[Sol] Letting $x = \tan\theta$, $\left(\text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$,

$$dx = \boxed{\frac{1}{\cos^2\theta}}\, d\theta$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{1}{\sqrt{\tan^2\theta+1}} \cdot \boxed{\frac{1}{\cos^2\theta}}\, d\theta$$

$$= \int \boxed{\frac{1}{\cos\theta}}\, d\theta$$

$\tan^2\theta + 1 = \frac{1}{\cos^2\theta}$

$$= \int \frac{\cos\theta}{\cos^2\theta} d\theta$$

$$= \int \frac{\cos\theta}{1-\sin^2\theta} d\theta$$

Letting $t = \sin\theta$, $\cos\theta d\theta = dt$

$$\therefore \int \frac{d\theta}{\cos\theta} = \int \frac{dt}{1-t^2}$$

$$= \frac{1}{2}\int \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt$$

$$= \frac{1}{2}(\ln|1+t| - \ln|1-t|) + C$$

$$= \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C$$

Since $\frac{1+t}{1-t} = \frac{1+\sin\theta}{1-\sin\theta} = \frac{(1+\sin\theta)^2}{1-\sin^2\theta} = \left(\frac{1+\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta} + \tan\theta\right)^2$

$= (\sqrt{x^2+1} + x)^2$

Therefore,

$$\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| = \ln(\sqrt{x^2+1} + x)$$

Thus, $\int \frac{1}{\sqrt{x^2+1}} dx = \ln(\sqrt{x^2+1} + x) + C$

Indefinite Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Evaluate each of the following indefinite integrals.

(1) $\displaystyle\int x\tan^2 x\,dx$

[Sol] $\displaystyle\int x\tan^2 x\,dx = \int x\left(\frac{1}{\cos^2 x} - 1\right)dx$

$$= \int \frac{x}{\cos^2 x}dx - \int x\,dx$$

Since $\displaystyle\int \frac{x}{\cos^2 x}dx = x\tan x - \int \tan x\,dx$

$$= x\tan x - \int \frac{(-\cos x)'}{\cos x}dx$$

$$= x\tan x + \ln|\cos x| + C$$

$$\therefore \int x\tan^2 x\,dx = x\tan x + \ln|\cos x| - \frac{x^2}{2} + C$$

(2) $\displaystyle\int x^3 e^{-x}dx = \int x^3(-e^{-x})'dx$

$$= -x^3e^{-x} + \int 3x^2e^{-x}dx$$

$$= -x^3e^{-x} + 3\left(-x^2e^{-x} + \int 2xe^{-x}dx\right)$$

$$= -x^3e^{-x} - 3x^2e^{-x} + 6\left(-xe^{-x} + \int e^{-x}dx\right)$$

$$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + C$$

$$[= -e^{-x}(x^3 + 3x^2 + 6x + 6) + C]$$

O 100 b

(3) $\displaystyle\int \frac{x^3-2x^2-1}{x^2-3x+2}dx$

[Sol] $\dfrac{x^3-2x^2-1}{x^2-3x+2} = x+1+\dfrac{x-3}{x^2-3x+2}$,

Letting $\dfrac{x-3}{x^2-3x+2} = \dfrac{P}{x-2}+\dfrac{Q}{x-1}$,

Determining the values of P and Q,

$$x-3 = P(x-1)+Q(x-2)$$
$$x-3 = x(P+Q)-(P+2Q) \qquad \cdots①$$

Matching the coefficients of the left- and right-hand sides of ①,

$$\begin{cases} P+Q=1 \\ P+2Q=3 \end{cases}$$

$\therefore P=-1, Q=2$

$$\therefore \int \frac{x^3-2x^2-1}{x^2-3x+2}dx = \int\left(x+1-\frac{1}{x-2}+\frac{2}{x-1}\right)dx$$

$$=\frac{1}{2}x^2+x-\ln|x-2|+2\ln|x-1|+C$$

$$\left[=\frac{1}{2}x^2+x+\ln\frac{(x-1)^2}{|x-2|}+C\right]$$

O 101 a Definite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	1	2	3~

Definition of Definite Integral

If $F(x)$ is the indefinite integral of $f(x)$, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Evaluate each of the following definite integrals.

(1) $\displaystyle\int_1^2 \sqrt{x^3}\,dx = \int_1^2 x^{\frac{3}{2}}\,dx$

$$= \left[\frac{2}{5}x^{\frac{5}{2}}\right]_1^2$$

$$= \frac{2}{5}\left(2^{\frac{5}{2}} - 1^{\frac{5}{2}}\right)$$

$$= \frac{2}{5}(4\sqrt{2} - 1)$$

(2) $\displaystyle\int_1^2 \frac{2x^2-1}{x}\,dx = \int_1^2 \left(2x - \frac{1}{x}\right)dx$

$$= \Big[\,x^2 - \ln|x|\,\Big]_1^2$$

$$= (4 - \ln 2) - (1 - \ln 1)$$

$$= 3 - \ln 2$$

(3) $\displaystyle\int_1^4 \frac{x+1}{\sqrt{x}}\,dx = \int_1^4 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right]_1^4$$

$$= \left(\frac{2}{3}\cdot 4^{\frac{3}{2}} + 2\cdot 4^{\frac{1}{2}}\right) - \left(\frac{2}{3} + 2\right)$$

$$= \frac{20}{3}$$

O 101 b

(4) $$\int_1^9 \frac{(1-\sqrt{x})^2}{x}dx = \int_1^9 \frac{1-2\sqrt{x}+x}{x}dx$$
$$= \int_1^9 \left(\frac{1}{x} - 2x^{-\frac{1}{2}} + 1\right)dx$$
$$= \left[\ln|x| - 4x^{\frac{1}{2}} + x\right]_1^9$$
$$= (\ln 9 - 12 + 9) - (\ln 1 - 4 + 1)$$
$$= 2\ln 3$$

(5) $$\int_0^2 \sqrt{2-x}\,dx = \left[-\frac{2}{3}(2-x)^{\frac{3}{2}}\right]_0^2$$
$$= \frac{2}{3}\cdot 2^{\frac{3}{2}}$$
$$= \frac{4}{3}\sqrt{2}$$

(6) $$\int_0^1 \frac{dx}{\sqrt{x}+\sqrt{x+1}} = \int_0^1 \frac{\sqrt{x}-\sqrt{x+1}}{(\sqrt{x}+\sqrt{x+1})(\sqrt{x}-\sqrt{x+1})}dx$$
$$= \int_0^1 (\sqrt{x+1}-\sqrt{x})dx$$
$$= \left[\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$
$$= \left(\frac{2}{3}\cdot 2^{\frac{3}{2}} - \frac{2}{3}\right) - \frac{2}{3} = \frac{4}{3}(\sqrt{2}-1)$$

(7) $$\int_0^4 \frac{dx}{\sqrt{x+5}-\sqrt{x}} = \int_0^4 \frac{\sqrt{x+5}+\sqrt{x}}{5}dx$$
$$= \frac{1}{5}\left[\frac{2}{3}(x+5)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$
$$= \frac{1}{5}\left[\left(\frac{2}{3}\cdot 9^{\frac{3}{2}} + \frac{2}{3}\cdot 4^{\frac{3}{2}}\right) - \frac{2}{3}\cdot 5^{\frac{3}{2}}\right]$$
$$= \frac{2}{15}(27+8-5\sqrt{5}) = \frac{14-2\sqrt{5}}{3}$$

Definite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	2	3-

Evaluate each of the following definite integrals.

(1) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta \, d\theta = \left[\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right]$$
$$= 1$$

(2) $$\int_0^{\frac{\pi}{2}} (2\cos x + \sin 2x)\, dx = \left[2\sin x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= \left(2\sin\frac{\pi}{2} - \frac{1}{2}\cos\pi\right) - \left(2\sin 0 - \frac{1}{2}\cos 0\right)$$
$$= 3$$

(3) $$\int_0^{\frac{\pi}{4}} \sin^2 x \, dx = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{2}\, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right)$$
$$= \frac{\pi}{8} - \frac{1}{4}$$

(4) $$\int_0^{\frac{\pi}{3}} (\sin x - \sin^2 x)\, dx = \int_0^{\frac{\pi}{3}} \left(\sin x - \frac{1-\cos 2x}{2}\right) dx$$
$$= \left[-\cos x - \frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{3}}$$
$$= \left(-\cos\frac{\pi}{3} - \frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3}\right) + \cos 0$$
$$= \frac{1}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

O 102 b

(5) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - 1\right) dx$$

$$= \Big[\tan x - x\Big]_0^{\frac{\pi}{4}}$$

$$= \tan\frac{\pi}{4} - \frac{\pi}{4}$$

$$= 1 - \frac{\pi}{4}$$

(6) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 3x \cos x\,dx$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}(\sin 4x + \sin 2x)\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\left(-\frac{1}{4}\cos\frac{4\pi}{3} - \frac{1}{2}\cos\frac{2\pi}{3}\right) - \left(-\frac{1}{4}\cos\frac{2\pi}{3} - \frac{1}{2}\cos\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{8} + \frac{1}{4} - \frac{1}{8} + \frac{1}{4}\right)$$

$$= \frac{1}{4}$$

(7) $\displaystyle\int_0^{\pi} (\sin x + \sin 2x)^2\,dx$

$$= \int_0^{\pi} (\sin^2 x + 2\sin x \sin 2x + \sin^2 2x)\,dx$$

$$= \int_0^{\pi} \left[\frac{1 - \cos 2x}{2} - (\cos 3x - \cos x) + \frac{1 - \cos 4x}{2}\right] dx$$

$$= \left[\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) - \left(\frac{\sin 3x}{3} - \sin x\right) + \frac{1}{2}\left(x - \frac{\sin 4x}{4}\right)\right]_0^{\pi}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

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O 103 a Definite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	2	3–

Evaluate each of the following definite integrals.

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2\left(x - \frac{\pi}{6}\right) dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[1 - \cos\left(2x - \frac{\pi}{3}\right)\right] dx$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin\left(2x - \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sqrt{3}}{4}\right) - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{\pi - \sqrt{3}}{4}$$

(2) $\displaystyle\int_0^{\frac{\pi}{2}} \sin\frac{5}{2}x\cos\frac{x}{2}\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin 3x + \sin 2x)\, dx$

$$= \frac{1}{2}\left[-\frac{1}{3}\cos 3x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{1}{2} - \left(-\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{2}{3}$$

(3) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x + \cos x)^2 dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1 + \sin 2x)\, dx$

$$= \left[x - \frac{1}{2}\cos 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\cos\frac{\pi}{2}\right) - \left(\frac{\pi}{6} - \frac{1}{2}\cos\frac{\pi}{3}\right)$$

$$= \frac{\pi}{12} + \frac{1}{4}$$

O 103 b

(4) $\displaystyle\int_1^3 (e^x - e^{-x})^2 dx = \int_1^3 (e^{2x} - 2 + e^{-2x}) dx$

$$= \left[\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right]_1^3$$

$$= \frac{1}{2}(e^6 - e^{-6} - e^2 + e^{-2}) - 4$$

(5) $\displaystyle\int_0^1 \frac{x^2+3x+1}{x+1} dx = \int_0^1 \left(x + 2 - \frac{1}{x+1}\right) dx$

$$= \left[\frac{1}{2}x^2 + 2x - \ln|x+1|\right]_0^1$$

$$= \frac{5}{2} - \ln 2$$

(6) $\displaystyle\int_0^1 \frac{dx}{x^2-4} = \frac{1}{4}\int_0^1 \left(\frac{1}{x-2} - \frac{1}{x+2}\right) dx$

$$= \frac{1}{4}\Big[\ln|x-2| - \ln|x+2|\Big]_0^1$$

$$= -\frac{1}{4}\ln 3$$

(7) $\displaystyle\int_{-1}^2 \frac{x}{x^2-x-6} dx = \int_{-1}^2 \frac{x}{(x-3)(x+2)} dx$

$$= \frac{1}{5}\int_{-1}^2 \left(\frac{3}{x-3} + \frac{2}{x+2}\right) dx$$

$$= \frac{1}{5}\Big[3\ln|x-3| + 2\ln|x+2|\Big]_{-1}^2$$

$$= -\frac{2}{5}\ln 2$$

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O 104 a

Definite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Evaluate each of the following definite integrals.

Ex. $\displaystyle\int_0^{\pi} |\cos x|\,dx$

[Sol] $f(x)=|\cos x|$ over $0 \leq x \leq \frac{\pi}{2}$ is the same as $f(x) = \cos x$, and

over $\frac{\pi}{2} \leq x \leq \pi$ is the same as $f(x) = -\cos x$.

$$\therefore \int_0^{\pi} |\cos x|\,dx = \int_0^{\frac{\pi}{2}} \cos x\,dx - \int_{\frac{\pi}{2}}^{\pi} \cos x\,dx$$

$$= \Big[\sin x\Big]_0^{\frac{\pi}{2}} - \Big[\sin x\Big]_{\frac{\pi}{2}}^{\pi} = 2$$

(1) $\displaystyle\int_0^{\pi} |\sin 2x|\,dx$

[Sol] $f(x) = |\sin 2x|$ over $0 \leq x \leq \frac{\pi}{2}$ is the same as $f(x) = \boxed{\sin 2x}$, and

over $\frac{\pi}{2} \leq x \leq \pi$ is the same as $f(x) = \boxed{-\sin 2x}$.

$$\therefore \int_0^{\pi} |\sin 2x|\,dx = \int_0^{\frac{\pi}{2}} \sin 2x\,dx - \int_{\frac{\pi}{2}}^{\pi} \sin 2x\,dx$$

$$= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} - \left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\pi} = 2$$

(2) $\displaystyle\int_0^{\pi} |\sin x + \cos x|\,dx$

[Sol] $\displaystyle\int_0^{\pi} |\sin x + \cos x|\,dx = \int_0^{\pi} \left|\sqrt{2}\sin\left(x + \boxed{\frac{\pi}{4}}\right)\right| dx$

$f(x) = \left|\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right|$ over $0 \leq x \leq \frac{3\pi}{4}$ is the same as $f(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$, and

over $\frac{3\pi}{4} \leq x \leq \pi$ is the same as $f(x) = -\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

$$\therefore \int_0^{\pi} |\sin x + \cos x|\,dx = \sqrt{2}\int_0^{\frac{3\pi}{4}} \sin\left(x + \frac{\pi}{4}\right)dx - \sqrt{2}\int_{\frac{3\pi}{4}}^{\pi} \sin\left(x + \frac{\pi}{4}\right)dx$$

$$= -\sqrt{2}\left[\cos\left(x + \frac{\pi}{4}\right)\right]_0^{\frac{3\pi}{4}} + \sqrt{2}\left[\cos\left(x + \frac{\pi}{4}\right)\right]_{\frac{3\pi}{4}}^{\pi}$$

$$= \sqrt{2} + 1 - 1 + \sqrt{2} = 2\sqrt{2}$$

O 105 b

(2) $\displaystyle\int_0^1 \frac{e^x}{(e^x+1)^2}dx$

[Sol] Letting $e^x + 1 = t, \quad e^x dx = dt$
When $x = 0, t = 2$; when $x = 1, t = e + 1$

$$\therefore \int_0^1 \frac{e^x}{(e^x+1)^2}dx = \int_2^{e+1} \frac{dt}{t^2}$$
$$= \left[-\frac{1}{t}\right]_2^{e+1}$$
$$= -\frac{1}{e+1} - \left(-\frac{1}{2}\right)$$
$$= \frac{e-1}{2(e+1)}$$

(3) $\displaystyle\int_0^\pi \frac{\sin x}{(2-\cos x)^3}dx$

[Sol] Letting $2 - \cos x = t, \quad \sin x\, dx = dt$
When $x = 0, t = 1$; when $x = \pi, t = 3$

$$\therefore \int_0^\pi \frac{\sin x}{(2-\cos x)^3}dx = \int_1^3 \frac{dt}{t^3}$$
$$= \int_1^3 t^{-3} dt$$
$$= -\frac{1}{2}\left[\frac{1}{t^2}\right]_1^3$$
$$= \frac{4}{9}$$

Definite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Using substitution, evaluate each of the following definite integrals.

(1) $\displaystyle\int_0^1 \frac{x}{x^2+1}dx$

[Sol] Letting $x^2+1=t$, $\quad 2x\,dx=dt \quad \therefore\ x\,dx=\frac{1}{2}dt$

When $x=0, t=1$; when $x=1, t=2$

$$\therefore \int_0^1 \frac{x}{x^2+1}dx=\int_1^2 \frac{1}{t}\cdot\frac{1}{2}dt$$

$$=\frac{1}{2}\Big[\ln|t|\Big]_1^2$$

$$=\frac{1}{2}\ln 2$$

(2) $\displaystyle\int_0^2 \frac{4e^x+8}{e^x+2x}dx$

[Sol] Letting $e^x+2x=t$, $\quad (e^x+2)dx=dt \quad \therefore\ (4e^x+8)dx=4dt$

When $x=0, t=1$; when $x=2, t=e^2+4$

$$\therefore \int_0^2 \frac{4e^x+8}{e^x+2x}dx=\int_1^{e^2+4} \frac{1}{t}\cdot 4dt$$

$$=4\Big[\ln|t|\Big]_1^{e^2+4}$$

$$=4\ln(e^2+4)$$

O 106 b

(3) $\displaystyle\int_1^2 \frac{x^3}{1+x^2}dx$

[Sol] Letting $1+x^2=t$, $\quad x^2=\boxed{t-1}$

Thus, $2x\,dx=dt \qquad \therefore\ x\,dx=\dfrac{1}{2}dt$

When $x=1, t=2$; when $x=2, t=5$

$$\therefore \int_1^2 \frac{x^3}{1+x^2}dx=\int_1^2 \frac{\boxed{x^2}}{1+x^2}\cdot x\,dx$$

$$=\int_2^5 \frac{t-1}{t}\cdot\frac{1}{2}dt$$

$$=\frac{1}{2}\int_2^5\left(1-\frac{1}{t}\right)dt$$

$$=\frac{1}{2}\Big[t-\ln|t|\Big]_2^5$$

$$=\frac{1}{2}(5-2)-\frac{1}{2}(\ln 5-\ln 2)$$

$$=\frac{3}{2}-\frac{1}{2}\ln\frac{5}{2}$$

(4) $\displaystyle\int_1^2 \frac{2x^3}{2x^2-1}dx$

[Sol] Letting $2x^2-1=t$, $\quad x^2=\dfrac{t+1}{2}$

Thus, $4x\,dx=dt \qquad \therefore\ x\,dx=\dfrac{1}{4}dt$

When $x=1, t=1$; when $x=2, t=7$

$$\therefore \int_1^2 \frac{2x^3}{2x^2-1}dx=\int_1^2 \frac{x^2}{2x^2-1}\cdot 2x\,dx$$

$$=\int_1^7 \frac{\frac{t+1}{2}}{t}\cdot\frac{1}{2}dt=\frac{1}{4}\int_1^7\frac{t+1}{t}dt$$

$$=\frac{1}{4}\Big[t+\ln|t|\Big]_1^7$$

$$=\frac{1}{4}(7-1)+\frac{1}{4}(\ln 7-\ln 1)$$

$$=\frac{3}{2}+\frac{1}{4}\ln 7$$

Definite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Using substitution, evaluate each of the following definite integrals.

(1) $\displaystyle\int_0^1 x\sqrt{1+x^2}\,dx$

[Sol] Letting $\sqrt{1+x^2}=t$, $\quad 1+x^2=t^2$

Thus, $2x\,dx=2t\,dt \quad \therefore x\,dx=t\,dt$

When $x=0$, $t=1$; when $x=1$, $t=\sqrt{2}$

$$\therefore \int_0^1 x\sqrt{1+x^2}\,dx=\int_1^{\sqrt{2}} t\cdot t\,dt$$

$$=\left[\frac{t^3}{3}\right]_1^{\sqrt{2}}$$

$$=\frac{2\sqrt{2}-1}{3}$$

(2) $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$

[Sol] Letting $\sqrt{x-1}=t$, $\quad x-1=t^2$

Thus, $dx=2t\,dt$

When $x=1$, $t=0$; when $x=2$, $t=1$

$$\therefore \int_1^2 x\sqrt{x-1}\,dx=\int_0^1 (t^2+1)t\cdot 2t\,dt$$

$$=2\int_0^1 (t^4+t^2)\,dt$$

$$=2\left[\frac{t^5}{5}+\frac{t^3}{3}\right]_0^1$$

$$=2\left(\frac{1}{5}+\frac{1}{3}\right)$$

$$=\frac{16}{15}$$

O 107 b

(3) $\displaystyle\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$

[Sol] Letting $1 - x^2 = t$, $\quad -2xdx = dt \quad \therefore\ xdx = -\frac{1}{2}dt$

When $x = 0, t = 1$; when $x = \frac{1}{2}, t = \frac{3}{4}$

$$\therefore \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{\frac{3}{4}} \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\int_{\frac{3}{4}}^1 \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]_{\frac{3}{4}}^1$$

$$= 1 - \frac{\sqrt{3}}{2}$$

$$= \frac{2-\sqrt{3}}{2}$$

(4) $\displaystyle\int_0^1 x^3\sqrt{1-x}\,dx$

[Sol] Letting $\sqrt{1-x} = t$, $\quad x = 1 - t^2$

Thus, $dx = -2tdt$

When $x = 0, t = 1$; when $x = 1, t = 0$

$$\therefore \int_0^1 x^3\sqrt{1-x}\,dx = \int_1^0 (1-t^2)^3 t \cdot (-2t)\,dt$$

$$= 2\int_0^1 (t^2 - 3t^4 + 3t^6 - t^8)\,dt$$

$$= 2\left[\frac{1}{3}t^3 - \frac{3}{5}t^5 + \frac{3}{7}t^7 - \frac{1}{9}t^9\right]_0^1$$

$$= \frac{32}{315}$$

Definite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

Using substitution, evaluate each of the following definite integrals.

(1) $\displaystyle\int_0^1 xe^{x^2}dx$

[Sol] Letting $x^2 = t$, $\quad 2x\,dx = dt$

When $x = 0, t = 0$; when $x = 1, t = 1$

$$\therefore \int_0^1 xe^{x^2}dx = \frac{1}{2}\int_0^1 e^t dt$$
$$= \frac{1}{2}\Big[e^t\Big]_0^1$$
$$= \frac{1}{2}(e-1)$$

(2) $\displaystyle\int_1^e \frac{\ln x}{x}dx$

[Sol] Letting $\ln x = t$, $\quad \dfrac{1}{x}dx = dt$

When $x = 1, t = 0$; when $x = e, t = 1$

$$\therefore \int_1^e \frac{\ln x}{x}dx = \int_0^1 t\,dt$$
$$= \left[\frac{t^2}{2}\right]_0^1$$
$$= \frac{1}{2}$$

(3) $\displaystyle\int_e^{e^2} \frac{dx}{x\ln x}$

[Sol] Letting $\ln x = t$, $\quad \dfrac{1}{x}dx = dt$

When $x = e, t = 1$; when $x = e^2, t = 2$

$$\therefore \int_e^{e^2} \frac{dx}{x\ln x} = \int_e^{e^2} \frac{1}{\ln x}\cdot\frac{1}{x}dx$$
$$= \int_1^2 \frac{dt}{t}$$
$$= \Big[\ln t\Big]_1^2 = \ln 2$$

O 108 b

(4) $\int_1^4 \frac{\sqrt{x}}{x-5}dx$

[Sol] Letting $\sqrt{x}=t$, $\quad x=t^2$

Thus, $dx=2t\,dt$

When $x=1, t=1$; when $x=4, t=2$

$$\therefore \int_1^4 \frac{\sqrt{x}}{x-5}dx=\int_1^2 \frac{t}{t^2-5}\cdot 2t\,dt$$

$$=2\int_1^2 \frac{t^2}{t^2-5}dt=2\int_1^2\left(1+\frac{5}{t^2-5}\right)dt$$

$$=2\int_1^2\left[1+\frac{\sqrt{5}}{2}\left(\frac{1}{t-\sqrt{5}}-\frac{1}{t+\sqrt{5}}\right)\right]dt$$

$$=\left[2t+\sqrt{5}\left(\ln|t-\sqrt{5}|-\ln|t+\sqrt{5}|\right)\right]_1^2$$

$$=\left[2t+\sqrt{5}\ln\left|\frac{t-\sqrt{5}}{t+\sqrt{5}}\right|\right]_1^2$$

$$=2+\sqrt{5}\left(\ln\frac{\sqrt{5}-2}{\sqrt{5}+2}-\ln\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)$$

$$=2+\sqrt{5}\ln\frac{(\sqrt{5}-2)(\sqrt{5}+1)}{(\sqrt{5}+2)(\sqrt{5}-1)}=2+\sqrt{5}\ln\frac{3-\sqrt{5}}{3+\sqrt{5}}$$

$$=2+\sqrt{5}\ln\frac{7-3\sqrt{5}}{2}$$

(5) $\int_0^1 xe^{1-x^2}dx$

[Sol] Letting $1-x^2=t$, $\quad -2x\,dx=dt \quad \therefore x\,dx=-\frac{1}{2}dt$

When $x=0, t=1$; when $x=1, t=0$

$$\therefore \int_0^1 xe^{1-x^2}dx=-\frac{1}{2}\int_1^0 e^t dt$$

$$=\frac{1}{2}\int_0^1 e^t dt$$

$$=\frac{1}{2}\left[e^t\right]_0^1$$

$$=\frac{1}{2}(e-1)$$

Definite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Using substitution, evaluate each of the following definite integrals.

(1) $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2 x \cos x \, dx$

[Sol] Letting $\sin x = t$, $\cos x \, dx = dt$

When $x = 0$, $t = 0$; when $x = \dfrac{\pi}{6}$, $t = \dfrac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x \cos x \, dx = \int_0^{\frac{1}{2}} t^2 \, dt$$

$$= \left[\frac{1}{3}t^3\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{24}$$

(2) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3 x \, dx$

[Sol] $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3 x \, dx = \int_0^{\frac{\pi}{4}} \sin^2 x \sin x \, dx = \int_0^{\frac{\pi}{4}} (1 - \cos^2 x) \sin x \, dx$

Letting $\cos x = t$, $-\sin x \, dx = dt$

When $x = 0$, $t = 1$; when $x = \dfrac{\pi}{4}$, $t = \dfrac{\sqrt{2}}{2}$

$$\therefore \int_0^{\frac{\pi}{4}} \sin^3 x \, dx = \int_1^{\frac{\sqrt{2}}{2}} (1 - t^2)(-dt)$$

$$= \int_1^{\frac{\sqrt{2}}{2}} (t^2 - 1) \, dt$$

$$= \left[\frac{1}{3}t^3 - t\right]_1^{\frac{\sqrt{2}}{2}}$$

$$= \frac{8 - 5\sqrt{2}}{12}$$

O 109 b

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{1+\cos x}\,dx$

[Sol] Letting $\cos x = t$, $\quad -\sin x\,dx = dt$

When $x = 0, t = 1$; when $x = \frac{\pi}{2}, t = 0$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{1+\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x(1-\cos^2 x)}{1+\cos x}\,dx$$
$$= \int_0^{\frac{\pi}{2}} (1-\cos x)\sin x\,dx$$
$$= -\int_1^0 (1-t)\,dt$$
$$= \int_1^0 (t-1)\,dt$$
$$= \left[\frac{1}{2}t^2 - t\right]_1^0$$
$$= \frac{1}{2}$$

(4) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{2-\sin^2 x}\,dx$

[Sol] Letting $\sin x = t$, $\quad \cos x\,dx = dt$

When $x = -\frac{\pi}{2}, t = -1$; when $x = \frac{\pi}{2}, t = 1$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{2-\sin^2 x}\,dx = \int_{-1}^{1} \frac{dt}{2-t^2} = -\frac{1}{2\sqrt{2}}\int_{-1}^{1}\left(\frac{1}{t-\sqrt{2}} - \frac{1}{t+\sqrt{2}}\right)dt$$
$$= -\frac{1}{2\sqrt{2}}\left[\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|\right]_{-1}^{1}$$
$$= -\frac{1}{2\sqrt{2}}\left(\ln\frac{\sqrt{2}-1}{\sqrt{2}+1} - \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$
$$= -\frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 = -\frac{1}{2\sqrt{2}}\ln(\sqrt{2}-1)^4$$
$$= -\sqrt{2}\ln(\sqrt{2}-1)$$
$$[= \sqrt{2}\ln(\sqrt{2}+1)]$$

O 110 a Definite Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Evaluate each of the following definite integrals.

(1) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}-\sqrt{x-1}} = \int_1^4 \frac{\sqrt{x}+\sqrt{x-1}}{(\sqrt{x}-\sqrt{x-1})(\sqrt{x}+\sqrt{x-1})}dx$

$$= \int_1^4 (\sqrt{x}+\sqrt{x-1})dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}}\right]_1^4$$

$$= \left(\frac{16}{3} + \frac{2}{3}\cdot 3\sqrt{3}\right) - \frac{2}{3} = \frac{14+6\sqrt{3}}{3}$$

(2) $\displaystyle\int_4^5 \frac{dx}{(x-1)(x-3)} = \frac{1}{2}\int_4^5 \left(\frac{1}{x-3} - \frac{1}{x-1}\right)dx$

$$= \frac{1}{2}\Big[\ln|x-3| - \ln|x-1|\Big]_4^5$$

$$= \frac{1}{2}\left[\ln\left|\frac{x-3}{x-1}\right|\right]_4^5$$

$$= \frac{1}{2}\left(\ln\frac{1}{2} - \ln\frac{1}{3}\right)$$

$$= \frac{1}{2}\ln\frac{3}{2}$$

O 110 b

(3) $\int_0^1 x^2\sqrt{1-x}\,dx$

[Sol] Letting $\sqrt{1-x}=t$, $\quad x=1-t^2$

Thus, $dx=-2t\,dt$

When $x=0, t=1$; when $x=1, t=0$

$$\therefore \int_0^1 x^2\sqrt{1-x}\,dx=\int_1^0 (1-t^2)^2\cdot t\cdot(-2t)\,dt$$

$$=2\int_0^1 (t^2-2t^4+t^6)\,dt$$

$$=2\left[\frac{1}{3}t^3-\frac{2}{5}t^5+\frac{1}{7}t^7\right]_0^1$$

$$=\frac{16}{105}$$

(4) $\int_0^{\frac{3\pi}{4}} \cos^3 x\,dx$

[Sol] $\int_0^{\frac{3\pi}{4}} \cos^3 x\,dx=\int_0^{\frac{3\pi}{4}} \cos^2 x\cos x\,dx=\int_0^{\frac{3\pi}{4}} (1-\sin^2 x)\cos x\,dx$

Letting $\sin x=t$, $\quad \cos x\,dx=dt$

When $x=0, t=0$; when $x=\frac{3\pi}{4}, t=\frac{\sqrt{2}}{2}$

$$\therefore \int_0^{\frac{3\pi}{4}} \cos^3 x\,dx=\int_0^{\frac{\sqrt{2}}{2}} (1-t^2)\,dt$$

$$=\left[t-\frac{1}{3}t^3\right]_0^{\frac{\sqrt{2}}{2}}$$

$$=\frac{5\sqrt{2}}{12}$$

Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

Given the formula for integration by parts for indefinite integrals,

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

we can derive the following formula for integration by parts for definite integrals:

$$\int_a^b f'(x)g(x)dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)g'(x)dx$$

Evaluate each of the following definite integrals.

Ex.

$$\int_0^1 x(x-3)^2 dx$$

$$= \left[\frac{x(x-3)^3}{3}\right]_0^1 - \frac{1}{3}\int_0^1 (x-3)^3 dx$$

$$= -\frac{8}{3} - \frac{1}{3}\left[\frac{(x-3)^4}{4}\right]_0^1$$

$$= \frac{11}{4}$$

(1) $\int_{-2}^{2} (x+2)x^2 dx$

$$= \left[\frac{x^3(x+2)}{3}\right]_{-2}^{2} - \frac{1}{3}\int_{-2}^{2} x^3 dx$$

$$= \frac{32}{3} - \frac{1}{3}\left[\frac{x^4}{4}\right]_{-2}^{2}$$

$$= \frac{32}{3}$$

O 111 b

(2) $\int_{-2}^{0}\frac{x}{(1-x)^4}dx$

$$=\int_{-2}^{0}x(1-x)^{-4}dx$$

$$=\frac{1}{3}\left[\frac{x}{(1-x)^3}\right]_{-2}^{0}-\frac{1}{3}\int_{-2}^{0}(1-x)^{-3}dx$$

$$=\frac{2}{81}-\frac{1}{6}\left[\frac{1}{(1-x)^2}\right]_{-2}^{0}$$

$$=-\frac{10}{81}$$

(3) $\int_{-2}^{-1}\frac{x^2}{(x+3)^4}dx$

$$=-\frac{1}{3}\left[\frac{x^2}{(x+3)^3}\right]_{-2}^{-1}+\frac{2}{3}\int_{-2}^{-1}\frac{x}{(x+3)^3}dx$$

$$=\frac{31}{24}+\frac{2}{3}\left\{-\frac{1}{2}\left[\frac{x}{(x+3)^2}\right]_{-2}^{-1}+\frac{1}{2}\int_{-2}^{-1}\frac{dx}{(x+3)^2}\right\}$$

$$=\frac{31}{24}-\frac{7}{12}+\frac{1}{3}\int_{-2}^{-1}\frac{dx}{(x+3)^2}$$

$$=\frac{17}{24}-\frac{1}{3}\left[\frac{1}{x+3}\right]_{-2}^{-1}$$

$$=\frac{17}{24}+\frac{1}{6}$$

$$=\frac{7}{8}$$

KUMON

O 112 a

Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69% –
(mistakes) 0	–	–	1	2–

Evaluate each of the following definite integrals.

(1) $\displaystyle\int_2^3 x\sqrt{x-2}\,dx$

$\displaystyle = \frac{2}{3}\Big[x(x-2)^{\frac{3}{2}}\Big]_2^3 - \frac{2}{3}\int_2^3 (x-2)^{\frac{3}{2}}dx$

$\displaystyle = 2 - \frac{4}{15}\Big[(x-2)^{\frac{5}{2}}\Big]_2^3$

$\displaystyle = \frac{26}{15}$

(2) $\displaystyle\int_{-\frac{1}{2}}^0 \frac{1}{2}x(2x+1)^{\frac{3}{2}}dx$

$\displaystyle = \frac{1}{10}\Big[x(2x+1)^{\frac{5}{2}}\Big]_{-\frac{1}{2}}^0 - \frac{1}{10}\int_{-\frac{1}{2}}^0 (2x+1)^{\frac{5}{2}}dx$

$\displaystyle = -\frac{1}{10}\int_{-\frac{1}{2}}^0 (2x+1)^{\frac{5}{2}}dx$

$\displaystyle = -\frac{1}{70}\Big[(2x+1)^{\frac{7}{2}}\Big]_{-\frac{1}{2}}^0$

$\displaystyle = -\frac{1}{70}$

O 112 b

(3) $\displaystyle\int_{-2}^{1} \frac{3x-2}{\sqrt{x+3}}\,dx$

$$= \int_{-2}^{1} (3x-2)(x+3)^{-\frac{1}{2}}\,dx$$

$$= 2\Big[(3x-2)(x+3)^{\frac{1}{2}}\Big]_{-2}^{1} - 6\int_{-2}^{1} (x+3)^{\frac{1}{2}}\,dx$$

$$= 20 - 4\Big[(x+3)^{\frac{3}{2}}\Big]_{-2}^{1}$$

$$= 20 - 28$$

$$= -8$$

(4) $\displaystyle\int_{1}^{\frac{3}{2}} \frac{2x^2}{(2x-1)\sqrt{2x-1}}\,dx$

$$= \int_{1}^{\frac{3}{2}} 2x^2(2x-1)^{-\frac{3}{2}}\,dx$$

$$= -2\left[\frac{x^2}{\sqrt{2x-1}}\right]_{1}^{\frac{3}{2}} + 4\int_{1}^{\frac{3}{2}} \frac{x}{\sqrt{2x-1}}\,dx$$

$$= \left(-\frac{9\sqrt{2}}{4} + 2\right) + \Big[4x\sqrt{2x-1}\Big]_{1}^{\frac{3}{2}} - 4\int_{1}^{\frac{3}{2}} \sqrt{2x-1}\,dx$$

$$= \left(-\frac{9\sqrt{2}}{4} + 2\right) + (6\sqrt{2} - 4) - \frac{4}{3}\Big[(2x-1)^{\frac{3}{2}}\Big]_{1}^{\frac{3}{2}}$$

$$= \left(-\frac{9\sqrt{2}}{4} + 2\right) + (6\sqrt{2} - 4) - \frac{4}{3}(2\sqrt{2} - 1)$$

$$= -\frac{2}{3} + \frac{13\sqrt{2}}{12}$$

Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

Evaluate each of the following definite integrals.

Ex.
$$\int_{-1}^{1} xe^x dx$$
$$= \left[xe^x\right]_{-1}^{1} - \int_{-1}^{1} e^x dx$$
$$= \left(e + \frac{1}{e}\right) - \left[e^x\right]_{-1}^{1}$$
$$= \left(e + \frac{1}{e}\right) - \left(e - \frac{1}{e}\right)$$
$$= \frac{2}{e}$$

(1) $\int_{0}^{1} e^x x^2 dx$

$$= \left[e^x x^2\right]_0^1 - \int_0^1 e^x(2x)dx$$
$$= e - 2\left[e^x x\right]_0^1 + 2\int_0^1 e^x dx$$
$$= e - 2e + 2\left[e^x\right]_0^1$$
$$= e - 2$$

O 113 b

(2) $\int_{-1}^{1} x^2 e^{1-x} dx$

$= \left[-x^2 e^{1-x}\right]_{-1}^{1} + 2\int_{-1}^{1} x e^{1-x} dx$

$= (-1+e^2) + 2\left[-xe^{1-x}\right]_{-1}^{1} + 2\int_{-1}^{1} e^{1-x} dx$

$= (-1+e^2) + 2(-1-e^2) + 2\left[-e^{1-x}\right]_{-1}^{1}$

$= (-1+e^2) + 2(-1-e^2) + 2(-1+e^2)$

$= e^2 - 5$

(3) $\int_0^1 x^3 e^{-x^2} dx$

[Sol] Letting $-x^2 = t$, $\quad -2xdx = dt$

When $x=0$, $t=0$; when $x=1$, $t=-1$

$\therefore \int_0^1 x^3 e^{-x^2} dx = \int_0^1 (-x^2) e^{-x^2} (-x) dx$

$= \int_0^{-1} te^t \cdot \frac{1}{2} dt$

$= \frac{1}{2}\int_0^{-1} te^t dt$

$= \frac{1}{2}\left[te^t\right]_0^{-1} - \frac{1}{2}\int_0^{-1} e^t dt$

$= -\frac{1}{2e} - \frac{1}{2}\left[e^t\right]_0^{-1}$

$= -\frac{1}{2e} - \frac{1}{2}\left(\frac{1}{e} - 1\right)$

$= -\frac{1}{e} + \frac{1}{2}$

KUMON

O 114 a

Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Evaluate each of the following definite integrals.

(1) $\int_1^e x\ln x\,dx$

$=\left[\frac{x^2}{2}\ln x\right]_1^e-\frac{1}{2}\int_1^e x\,dx$

$=\frac{e^2}{2}-\left[\frac{x^2}{4}\right]_1^e$

$=\frac{1}{4}(e^2+1)$

(2) $\int_e^{e^2} x(\ln x)^2\,dx$

$=\frac{1}{2}\left[x^2(\ln x)^2\right]_e^{e^2}-\frac{1}{2}\int_e^{e^2} x^2(2\ln x)\cdot\frac{1}{x}\,dx$

$=\left(2e^4-\frac{1}{2}e^2\right)-\int_e^{e^2} x(\ln x)\,dx$

$=\left(2e^4-\frac{1}{2}e^2\right)-\frac{1}{2}\left[x^2\ln x\right]_e^{e^2}+\frac{1}{2}\int_e^{e^2} x\,dx$

$=\left(2e^4-\frac{1}{2}e^2\right)-\left(e^4-\frac{1}{2}e^2\right)+\frac{1}{4}\left[x^2\right]_e^{e^2}$

$=\left(2e^4-\frac{1}{2}e^2\right)-\left(e^4-\frac{1}{2}e^2\right)+\frac{1}{4}(e^4-e^2)$

$=\frac{5}{4}e^4-\frac{1}{4}e^2$

O 114 b

(3) $\int_1^e 4x^3(\ln x)^3 dx$

$$= \left[x^4(\ln x)^3\right]_1^e - 3\int_1^e x^3(\ln x)^2 dx$$

$$= e^4 - \frac{3}{4}\left[x^4(\ln x)^2\right]_1^e + \frac{3}{2}\int_1^e x^3 \ln x\, dx$$

$$= e^4 - \frac{3}{4}e^4 + \frac{3}{8}\left[x^4 \ln x\right]_1^e - \frac{3}{8}\int_1^e x^3 dx$$

$$= \frac{1}{4}e^4 + \frac{3}{8}e^4 - \frac{3}{32}\left[x^4\right]_1^e$$

$$= \frac{5}{8}e^4 - \frac{3}{32}e^4 + \frac{3}{32}$$

$$= \frac{17}{32}e^4 + \frac{3}{32}$$

O 115 a

Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	-	-	1	2-

Evaluate each of the following definite integrals.

(1) $\int_1^2 \frac{\ln x}{x^2} dx$

$= \left[-\frac{1}{x}\ln x\right]_1^2 - \int_1^2\left(-\frac{1}{x}\right)\cdot\frac{1}{x}dx$

$= -\frac{1}{2}\ln 2 + \left[-\frac{1}{x}\right]_1^2$

$= -\frac{1}{2}\ln 2 - \frac{1}{2} + 1$

$= \frac{1}{2}(1 - \ln 2)$

(2) $\int_1^3 \frac{1-\ln x}{x^2} dx$

$= \left[-\frac{1}{x}\cdot(1-\ln x)\right]_1^3 - \int_1^3\left(-\frac{1}{x}\right)\cdot\left(-\frac{1}{x}\right)dx$

$= \left[-\frac{1-\ln x}{x}\right]_1^3 - \int_1^3\frac{dx}{x^2}$

$= \left(\frac{1}{3}\ln 3 + \frac{2}{3}\right) + \left[\frac{1}{x}\right]_1^3$

$= \left(\frac{1}{3}\ln 3 + \frac{2}{3}\right) + \left(\frac{1}{3} - 1\right)$

$= \frac{1}{3}\ln 3$

O 115 b

(3) $\displaystyle\int_e^{e^2} \frac{\ln x}{x} dx$

$\displaystyle = \Big[(\ln x)^2\Big]_e^{e^2} - \int_e^{e^2} \frac{\ln x}{x} dx$

$\displaystyle = 3 - \int_e^{e^2} \frac{\ln x}{x} dx$

Rearranging.

$\displaystyle 2\int_e^{e^2} \frac{\ln x}{x} dx = 3$

$\displaystyle \therefore \int_e^{e^2} \frac{\ln x}{x} dx = \frac{3}{2}$

(4) $\displaystyle\int_e^{3} \frac{\ln x^2}{2x^2} dx$

$\displaystyle = -\frac{1}{2}\left[\frac{\ln x^2}{x}\right]_e^3 + \int_e^3 \frac{dx}{x^2}$

$\displaystyle = -\frac{1}{2}\left(\frac{\ln 9}{3} - \frac{2}{e}\right) - \left[\frac{1}{x}\right]_e^3$

$\displaystyle = -\frac{\ln 9}{6} + \frac{1}{e} - \frac{1}{3} + \frac{1}{e}$

$\displaystyle = -\frac{\ln 3}{3} - \frac{1}{3} + \frac{2}{e}$

O 116 a — Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Evaluate each of the following definite integrals.

Ex.

$$\int_0^{\frac{\pi}{2}} x\sin x\,dx$$
$$= -\Big[x\cos x\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos x\,dx$$
$$= \Big[\sin x\Big]_0^{\frac{\pi}{2}}$$
$$= 1$$

(1) $\displaystyle\int_0^{\frac{\pi}{3}} x\sin 2x\,dx$

$$= -\frac{1}{2}\Big[x\cos 2x\Big]_0^{\frac{\pi}{3}} + \frac{1}{2}\int_0^{\frac{\pi}{3}}\cos 2x\,dx$$
$$= \frac{\pi}{12} + \frac{1}{4}\Big[\sin 2x\Big]_0^{\frac{\pi}{3}}$$
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

(2) $\displaystyle\int_{-\frac{\pi}{4}}^{0} x\cos x\,dx$

$$= \Big[x\sin x\Big]_{-\frac{\pi}{4}}^{0} - \int_{-\frac{\pi}{4}}^{0}\sin x\,dx$$
$$= -\frac{\pi\sqrt{2}}{8} + \Big[\cos x\Big]_{-\frac{\pi}{4}}^{0}$$
$$= -\frac{\pi\sqrt{2}}{8} + 1 - \frac{\sqrt{2}}{2}$$

O 116 b

(3) $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} x\cos 3x\,dx$

$\displaystyle=\frac{1}{3}\Big[x\sin 3x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}-\frac{1}{3}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\sin 3x\,dx$

$\displaystyle=\left(\frac{\pi\sqrt{2}}{8}+\frac{\pi}{6}\right)+\frac{1}{9}\Big[\cos 3x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}$

$\displaystyle=\frac{\pi\sqrt{2}}{8}+\frac{\pi}{6}+\frac{\sqrt{2}}{18}$

(4) $\displaystyle\int_{0}^{\frac{\pi}{2}} x^2\sin x\,dx$

$\displaystyle=\Big[-x^2\cos x\Big]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}2x\cos x\,dx$

$\displaystyle=0+2\int_0^{\frac{\pi}{2}}x\cos x\,dx$

$\displaystyle=2\Big[x\sin x\Big]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\sin x\,dx$

$\displaystyle=\pi+2\Big[\cos x\Big]_0^{\frac{\pi}{2}}$

$=\pi-2$

KUMON

O 117 a

Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Evaluate each of the following definite integrals.

(1) $\int_{-\frac{\pi}{4}}^{0} x^2 \cos x \, dx$

$= \left[x^2 \sin x \right]_{-\frac{\pi}{4}}^{0} - 2\int_{-\frac{\pi}{4}}^{0} x \sin x \, dx$

$= \frac{\pi^2\sqrt{2}}{32} + 2\left[x \cos x \right]_{-\frac{\pi}{4}}^{0} - 2\int_{-\frac{\pi}{4}}^{0} \cos x \, dx$

$= \frac{\pi^2\sqrt{2}}{32} + \frac{\pi\sqrt{2}}{4} - 2\left[\sin x \right]_{-\frac{\pi}{4}}^{0}$

$= \frac{\pi^2\sqrt{2}}{32} + \frac{\pi\sqrt{2}}{4} - \sqrt{2}$

(2) $\int_{0}^{\frac{\pi}{4}} \frac{x}{\cos^2 x} dx$

$= \left[x \tan x \right]_{0}^{\frac{\pi}{4}} - \int_{0}^{\frac{\pi}{4}} \tan x \, dx$

$= \frac{\pi}{4} - \int_{0}^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx \qquad \left[= \frac{\pi}{4} - \int_{0}^{\frac{\pi}{4}} \frac{(-\cos x)'}{\cos x} dx \right]$

$= \frac{\pi}{4} + \left[\ln|\cos x| \right]_{0}^{\frac{\pi}{4}}$

$= \frac{\pi}{4} + \ln\frac{\sqrt{2}}{2}$

$\left[= \frac{\pi}{4} - \ln\sqrt{2} \right]$

O 117 b

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x\, dx$

$$= \int_0^{\frac{\pi}{2}} \sin^3 x \sin x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^3 x(-\cos x)'\, dx$$

$$= -\Big[\sin^3 x \cos x\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (\sin^3 x)' \cos x\, dx$$

$$= 0 + 3\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x\, dx = 3\int_0^{\frac{\pi}{2}} (\sin x \cos x)^2\, dx$$

$$= \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 2x\, dx$$

$$= \frac{3}{8}\int_0^{\frac{\pi}{2}} (1 - \cos 4x)\, dx = \frac{3}{8}\Big[x - \frac{1}{4}\sin 4x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{16}\pi$$

(4) $\displaystyle\int_{-\pi}^{\pi} 2\cos^4 x\, dx$

$$= 2\int_{-\pi}^{\pi} \cos^3 x \cos x\, dx$$

$$= 2\int_{-\pi}^{\pi} \cos^3 x(\sin x)'\, dx$$

$$= 2\Big[\sin x \cos^3 x\Big]_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi} (\cos^3 x)' \sin x\, dx$$

$$= 0 + 6\int_{-\pi}^{\pi} \sin^2 x \cos^2 x\, dx = 6\int_{-\pi}^{\pi} (\sin x \cos x)^2\, dx$$

$$= \frac{3}{2}\int_{-\pi}^{\pi} \sin^2 2x\, dx$$

$$= \frac{3}{4}\int_{-\pi}^{\pi} (1 - \cos 4x)\, dx = \frac{3}{4}\Big[x - \frac{1}{4}\sin 4x\Big]_{-\pi}^{\pi}$$

$$= \frac{3}{2}\pi$$

KUMON

O 118 a Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Evaluate each of the following definite integrals.

(1) $\displaystyle\int_0^{\frac{\pi}{2}} e^x \sin x\,dx$

[Sol] Letting $\displaystyle\int_0^{\frac{\pi}{2}} e^x \sin x\,dx = I$,

$$I = \Big[e^x \sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x\,dx$$

$$= e^{\frac{\pi}{2}} - \Big[e^x \cos x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x\,dx$$

$$= e^{\frac{\pi}{2}} + 1 - I$$

$$\therefore I = \frac{e^{\frac{\pi}{2}} + 1}{2}$$

(2) $\displaystyle\int_0^{\frac{\pi}{4}} e^x \cos x\,dx$

[Sol] Letting $\displaystyle\int_0^{\frac{\pi}{4}} e^x \cos x\,dx = I$,

$$I = \Big[e^x \cos x\Big]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} e^x \sin x\,dx$$

$$= \left(\frac{e^{\frac{\pi}{4}}\sqrt{2}}{2} - 1\right) + \Big[e^x \sin x\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x \cos x\,dx$$

$$= e^{\frac{\pi}{4}}\sqrt{2} - 1 - I$$

$$\therefore I = \frac{e^{\frac{\pi}{4}}\sqrt{2} - 1}{2}$$

O 118 b

(3) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} e^{2x}\sin 4x\,dx$

[Sol] Letting $\displaystyle\int_{\frac{\pi}{2}}^{\pi} e^{2x}\sin 4x\,dx = I.$

$$I = \frac{1}{2}\left[e^{2x}\sin 4x\right]_{\frac{\pi}{2}}^{\pi} - 2\int_{\frac{\pi}{2}}^{\pi} e^{2x}\cos 4x\,dx$$

$$= 0 - 2\int_{\frac{\pi}{2}}^{\pi} e^{2x}\cos 4x\,dx$$

$$= -\left[e^{2x}\cos 4x\right]_{\frac{\pi}{2}}^{\pi} - 4\int_{\frac{\pi}{2}}^{\pi} e^{2x}\sin 4x\,dx$$

$$= e^{\pi} - e^{2\pi} - 4I$$

$$\therefore I = \frac{e^{\pi} - e^{2\pi}}{5}$$

(4) $\displaystyle\int_{0}^{\pi} e^{2x}\sin x\,dx$

[Sol] Letting $\displaystyle\int_{0}^{\pi} e^{2x}\sin x\,dx = I.$

$$I = \frac{1}{2}\left[e^{2x}\sin x\right]_{0}^{\pi} - \frac{1}{2}\int_{0}^{\pi} e^{2x}\cos x\,dx$$

$$= -\frac{1}{4}\left[e^{2x}\cos x\right]_{0}^{\pi} - \frac{1}{4}\int_{0}^{\pi} e^{2x}\sin x\,dx$$

$$= \frac{1}{4}e^{2\pi} + \frac{1}{4} - \frac{1}{4}I$$

$$\therefore I = \frac{e^{2\pi} + 1}{5}$$

Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Evaluate each of the following definite integrals.

(1) $\displaystyle\int_0^{\frac{a}{2}} \sqrt{a^2 - x^2}\,dx \qquad (a > 0)$

[Sol] Letting $x = a\sin\theta \quad \left(-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}\right)$, $\quad \dfrac{dx}{d\theta} = a \cdot \boxed{\cos\theta}$ ... ①

When $x = 0$, $\theta = 0$; when $x = \dfrac{a}{2}$, $\theta = \boxed{\dfrac{\pi}{6}}$

Since $x = a\sin\theta$, $\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin\theta)^2} = \boxed{a \cdot \cos\theta}$ ... ②

Using ① and ②,

$$\int_0^{\frac{a}{2}} \sqrt{a^2 - x^2}\,dx = \int_0^{\frac{\pi}{6}} (a^2\cos^2\theta)\,d\theta = \frac{a^2}{2}\int_0^{\frac{\pi}{6}} (1 + \cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} = \left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)a^2$$

(2) $\displaystyle\int_0^{\sqrt{2}} \frac{dx}{\sqrt{4 - x^2}}$

[Sol] Letting $x = 2\sin\theta \quad \left(-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}\right)$, $\quad \dfrac{dx}{d\theta} = 2\cos\theta$ ... ①

When $x = 0$, $\theta = 0$; when $x = \sqrt{2}$, $\theta = \dfrac{\pi}{4}$

Since $x = 2\sin\theta$, $\sqrt{4 - x^2} = \sqrt{4 - (2\sin\theta)^2} = 2\cos\theta$. ... ②

Using ① and ②,

$$\int_0^{\sqrt{2}} \frac{dx}{\sqrt{4 - x^2}} = \int_0^{\frac{\pi}{4}} \frac{1}{2\cos\theta} \cdot 2\cos\theta\,d\theta = \int_0^{\frac{\pi}{4}} d\theta$$

$$= \left[\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

O 119 b

(3) $\displaystyle\int_0^2 \frac{dx}{x^2+4}$

[Sol] Letting $x = 2\tan\theta \quad \left(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\right)$, $\quad \dfrac{dx}{d\theta} = \dfrac{2}{\cos^2\theta}$ ...①

When $x = 0$, $\theta = 0$; when $x = 2$, $\theta = \dfrac{\pi}{4}$

Since $x = 2\tan\theta$, $x^2 + 4 = 4(\tan^2\theta + 1) = \dfrac{4}{\cos^2\theta}$ ...②

Using ① and ②,

$$\int_0^2 \frac{dx}{x^2+4} = \int_0^{\frac{\pi}{4}} \frac{\dfrac{2}{\cos^2\theta}}{\dfrac{4}{\cos^2\theta}}\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} d\theta$$

$$= \frac{1}{2}\Big[\theta\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{8}$$

(4) $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{3+x^2}$

[Sol] Letting $x = \sqrt{3}\tan\theta \quad \left(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\right)$, $\quad \dfrac{dx}{d\theta} = \dfrac{\sqrt{3}}{\cos^2\theta}$ ...①

When $x = 1$, $\theta = \dfrac{\pi}{6}$; when $x = \sqrt{3}$, $\theta = \dfrac{\pi}{4}$

Since $x = \sqrt{3}\tan\theta$, $3 + x^2 = 3(1 + \tan^2\theta) = \dfrac{3}{\cos^2\theta}$ ...②

Using ① and ②,

$$\int_1^{\sqrt{3}} \frac{dx}{3+x^2} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\dfrac{\sqrt{3}}{\cos^2\theta}}{\dfrac{3}{\cos^2\theta}}\, d\theta = \frac{\sqrt{3}}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta$$

$$= \frac{\sqrt{3}}{3}\Big[\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{\sqrt{3}}{36}\pi$$

O 120 a Definite Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

Evaluate each of the following definite integrals.

(1) $\int_{-1}^{1}(2x+5)x^2dx$

$=\frac{1}{3}\Big[x^3(2x+5)\Big]_{-1}^{1}-\frac{2}{3}\int_{-1}^{1}x^3dx$

$=\frac{10}{3}-\frac{1}{6}\Big[x^4\Big]_{-1}^{1}$

$=\frac{10}{3}$

(2) $\int_{2}^{3}2x^2e^{x-2}dx$

$=2\Big[x^2(e^{x-2})\Big]_{2}^{3}-4\int_{2}^{3}xe^{x-2}dx$

$=18e-8-4\Big[xe^{x-2}\Big]_{2}^{3}+4\int_{2}^{3}e^{x-2}dx$

$=6e+4\Big[e^{x-2}\Big]_{2}^{3}$

$=10e-4$

O 120 b

(3) $\displaystyle\int_1^2 9x^2(\ln x)^2 dx$

$\displaystyle = 3\Big[x^3(\ln x)^2\Big]_1^2 - 6\int_1^2 x^2 \ln x\, dx$

$\displaystyle = 24(\ln 2)^2 - 2\Big[x^3 \ln x\Big]_1^2 + 2\int_1^2 x^2 dx$

$\displaystyle = 24(\ln 2)^2 - 16\ln 2 + \frac{2}{3}\Big[x^3\Big]_1^2$

$\displaystyle = 24(\ln 2)^2 - 16\ln 2 + \frac{14}{3}$

(4) $\displaystyle\int_0^{\frac{\pi}{4}} x\cos 2x\, dx$

$\displaystyle = \frac{1}{2}\Big[x\sin 2x\Big]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 2x\, dx$

$\displaystyle = \frac{\pi}{8} + \frac{1}{4}\Big[\cos 2x\Big]_0^{\frac{\pi}{4}}$

$\displaystyle = \frac{\pi}{8} - \frac{1}{4}$

(5) $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\sin x\, dx$

[Sol] Letting $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\sin x\, dx = I$,

$\displaystyle I = \frac{1}{2}\Big[e^{2x}\sin x\Big]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{2x}\cos x\, dx$

$\displaystyle = \frac{1}{2}e^{\pi} - \frac{1}{4}\Big[e^{2x}\cos x\Big]_0^{\frac{\pi}{2}} - \frac{1}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin x\, dx$

$\displaystyle = \frac{1}{2}e^{\pi} + \frac{1}{4} - \frac{1}{4}I$

$\displaystyle \therefore I = \frac{1}{5}(2e^{\pi} + 1)$

KUMON

O 121 a Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

Evaluate each of the following integrals.

(1) $\int \frac{dx}{x-2} = \ln|x-2| + C$

(2) $\int \frac{2}{2x-3}dx = \ln|2x-3| + C$

(3) $\int \frac{3}{4x+1}dx = \frac{3}{4}\ln|4x+1| + C$

(4) $\int \frac{2x}{x^2+1}dx = \ln|x^2+1| + C$

$[= \ln(x^2+1) + C]$

O 121 b

(5) $\displaystyle\int \frac{2}{x^2+3x-4}dx$

[Sol] $$\frac{2}{x^2+3x-4}=\frac{2}{(x-1)(x+4)}$$

Letting $\dfrac{2}{(x-1)(x+4)}=\dfrac{P}{x-1}+\dfrac{Q}{x+4}$.

Determining the values of P and Q.

$$2=P(x+4)+Q(x-1) \quad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①.

$$\begin{cases} P+Q=0 \\ 4P-Q=2 \end{cases} \qquad \therefore P=\frac{2}{5},\ Q=-\frac{2}{5}$$

$$\begin{aligned} \therefore \int \frac{2}{x^2+3x-4}dx &= \int \left[\frac{2}{5(x-1)}-\frac{2}{5(x+4)}\right]dx \\ &= \frac{2}{5}\ln|x-1|-\frac{2}{5}\ln|x+4|+C \\ &\left[=\frac{2}{5}\ln\left|\frac{x-1}{x+4}\right|+C\right] \end{aligned}$$

(6) $\displaystyle\int \frac{dx}{x^2-a^2}$

[Sol] $$\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}$$

Letting $\dfrac{1}{(x-a)(x+a)}=\dfrac{P}{x-a}+\dfrac{Q}{x+a}$.

Determining the values of P and Q.

$$1=P(x+a)+Q(x-a) \quad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①.

$$\begin{cases} P+Q=0 \\ (P-Q)a=1 \end{cases} \qquad \therefore P=\frac{1}{2a},\ Q=-\frac{1}{2a}$$

$$\begin{aligned} \therefore \int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx \\ &= \frac{1}{2a}(\ln|x-a|-\ln|x+a|)+C \\ &\left[=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C\right] \end{aligned}$$

O 122 a Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2-

Evaluate each of the following integrals.

(1) $\displaystyle\int \frac{dx}{x(x+1)(x+2)}$

[Sol] Letting $\dfrac{1}{x(x+1)(x+2)} = \dfrac{P}{x} + \dfrac{Q}{x+1} + \dfrac{R}{x+2}$,

Determining the values of P, Q, and R,

$$1 = P(x+1)(x+2) + Qx(x+2) + Rx(x+1) \quad \text{—①}$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P+Q+R=0 \\ 3P+2Q+R=0 \\ 2P=1 \end{cases} \qquad \therefore P=\frac{1}{2},\quad Q=-1,\quad R=\frac{1}{2}$$

$$\therefore \int \frac{dx}{x(x+1)(x+2)} = \int\left[\frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}\right]dx$$

$$= \frac{1}{2}\ln|x| - \ln|x+1| + \frac{1}{2}\ln|x+2| + C$$

$$\left[= \frac{1}{2}\ln\frac{|x(x+2)|}{(x+1)^2} + C\right]$$

(2) $\displaystyle\int \frac{6}{x(x-3)(x+2)}dx$

[Sol] Letting $\dfrac{6}{x(x-3)(x+2)} = \dfrac{P}{x} + \dfrac{Q}{x-3} + \dfrac{R}{x+2}$,

Determining the values of P, Q, and R,

$$6 = P(x-3)(x+2) + Qx(x+2) + Rx(x-3) \quad \text{—①}$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P+Q+R=0 \\ -P+2Q-3R=0 \\ -6P=6 \end{cases} \qquad \therefore P=-1,\quad Q=\frac{2}{5},\quad R=\frac{3}{5}$$

$$\therefore \int \frac{6}{x(x-3)(x+2)}dx = \int\left[-\frac{1}{x} + \frac{2}{5(x-3)} + \frac{3}{5(x+2)}\right]dx$$

$$= -\ln|x| + \frac{2}{5}\ln|x-3| + \frac{3}{5}\ln|x+2| + C$$

$$\left[= \frac{1}{5}\ln\left|\frac{(x-3)^2(x+2)^3}{x^5}\right| + C\right]$$

O 122 b

(3) $\displaystyle\int \frac{4x^2+16x-8}{x^3-4x}\,dx$

[Sol] Letting $\dfrac{4x^2+16x-8}{x^3-4x}=\dfrac{P}{x}+\dfrac{Q}{x-2}+\dfrac{R}{x+2}$,

Determining the values of P, Q, and R,

$$4x^2+16x-8=P(x-2)(x+2)+Qx(x+2)+Rx(x-2) \qquad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P+Q+R=4 \\ 2Q-2R=16 \\ -4P=-8 \end{cases} \qquad \therefore P=2,\quad Q=5,\quad R=-3$$

$$\begin{aligned} \therefore \int \frac{4x^2+16x-8}{x^3-4x}\,dx &= \int\left(\frac{2}{x}+\frac{5}{x-2}-\frac{3}{x+2}\right)dx \\ &= 2\ln|x|+5\ln|x-2|-3\ln|x+2|+C \\ &\left[=\ln\left|\frac{x^2(x-2)^5}{(x+2)^3}\right|+C\right] \end{aligned}$$

(4) $\displaystyle\int \frac{x^2+8x+2}{6x^3+x^2-11x-6}\,dx$

[Sol] Letting $\dfrac{x^2+8x+2}{6x^3+x^2-11x-6}=\dfrac{P}{x+1}+\dfrac{Q}{3x+2}+\dfrac{R}{2x-3}$,

Determining the values of P, Q, and R,

$$x^2+8x+2=P(3x+2)(2x-3)+Q(x+1)(2x-3)+R(x+1)(3x+2) \qquad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} 6P+2Q+3R=1 \\ -5P-Q+5R=8 \\ -6P-3Q+2R=2 \end{cases} \qquad \therefore P=-1,\quad Q=2,\quad R=1$$

$$\begin{aligned} \therefore \int \frac{x^2+8x+2}{6x^3+x^2-11x-6}\,dx &= \int\left(-\frac{1}{x+1}+\frac{2}{3x+2}+\frac{1}{2x-3}\right)dx \\ &= -\ln|x+1|+\frac{2}{3}\ln|3x+2|+\frac{1}{2}\ln|2x-3|+C \\ &\left[=\frac{1}{6}\ln\frac{|(3x+2)^4(2x-3)^3|}{(x+1)^6}+C\right] \end{aligned}$$

O 123 a Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Evaluate each of the following integrals.

(1) $\displaystyle\int \frac{4x^2+3}{(2x+1)^2}dx$

[Sol] Dividing, $\dfrac{4x^2+3}{(2x+1)^2} = 1 - \dfrac{4x-2}{(2x+1)^2}$

Letting $\dfrac{4x-2}{(2x+1)^2} = \dfrac{P}{2x+1} + \dfrac{Q}{(2x+1)^2}$,

Determining the values of P and Q,

$$4x-2 = P(2x+1)+Q \quad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} 2P = 4 \\ P+Q = -2 \end{cases} \quad \therefore P=2,\ Q=-4$$

$$\therefore \int \frac{4x^2+3}{(2x+1)^2}dx = \int \left[1 - \frac{2}{2x+1} + \frac{4}{(2x+1)^2}\right]dx$$

$$= x - \ln|2x+1| - \frac{2}{2x+1} + C$$

(2) $\displaystyle\int \frac{8(x+1)^2}{(2x+3)^2}dx$

[Sol] Dividing, $\dfrac{8(x+1)^2}{(2x+3)^2} = 2 - \dfrac{8x+10}{(2x+3)^2}$

Letting $\dfrac{8x+10}{(2x+3)^2} = \dfrac{P}{2x+3} + \dfrac{Q}{(2x+3)^2}$,

Determining the values of P and Q,

$$8x+10 = P(2x+3)+Q \quad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} 2P = 8 \\ 3P+Q = 10 \end{cases} \quad \therefore P=4,\ Q=-2$$

$$\therefore \int \frac{8(x+1)^2}{(2x+3)^2}dx = \int \left[2 - \frac{4}{2x+3} + \frac{2}{(2x+3)^2}\right]dx$$

$$= 2x - 2\ln|2x+3| - \frac{1}{2x+3} + C$$

(3) $\displaystyle\int\frac{-dx}{(x+2)(3x+5)^2}$

[Sol] Letting $\dfrac{-1}{(x+2)(3x+5)^2}=\dfrac{P}{x+2}+\dfrac{Q}{3x+5}+\dfrac{R}{(3x+5)^2}$,

Determining the values of P, Q, and R,

$$-1=P(3x+5)^2+Q(x+2)(3x+5)+R(x+2) \quad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} 9P+3Q=0 \\ 30P+11Q+R=0 \\ 25P+10Q+2R=-1 \end{cases} \qquad \therefore P=-1,\quad Q=3,\quad R=-3$$

$$\therefore \int\frac{-dx}{(x+2)(3x+5)^2}=\int\left[-\frac{1}{x+2}+\frac{3}{3x+5}-\frac{3}{(3x+5)^2}\right]dx$$

$$=-\ln|x+2|+\ln|3x+5|+\frac{1}{3x+5}+C$$

$$\left[=\ln\left|\frac{3x+5}{x+2}\right|+\frac{1}{3x+5}+C\right]$$

(4) $\displaystyle\int\frac{x^5-x^3+1}{x^4-x^3}dx$

[Sol] Dividing, $\dfrac{x^5-x^3+1}{x^4-x^3}=x+1+\dfrac{1}{x^3(x-1)}$

Letting $\dfrac{1}{x^3(x-1)}=\dfrac{P}{x-1}+\dfrac{Q}{x}+\dfrac{R}{x^2}+\dfrac{S}{x^3}$,

Determining the values of P, Q, R, and S,

$$1=Px^3+Qx^2(x-1)+Rx(x-1)+S(x-1) \quad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P+Q=0 \\ -Q+R=0 \\ -R+S=0 \\ -S=1 \end{cases} \qquad \therefore P=1,\quad Q=R=S=-1$$

$$\therefore \int\frac{x^5-x^3+1}{x^4-x^3}dx=\int\left(x+1+\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}\right)dx$$

$$=\frac{1}{2}x^2+x+\ln|x-1|-\ln|x|+\frac{1}{x}+\frac{1}{2x^2}+C$$

$$\left[=\frac{1}{2}x^2+x+\ln\left|\frac{x-1}{x}\right|+\frac{1}{x}+\frac{1}{2x^2}+C\right]$$

O 124 a

Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Evaluate each of the following integrals.

(1) $\displaystyle\int \frac{x(x^2+1)}{x^3+1}dx$

[Sol] Dividing, $\displaystyle\frac{x(x^2+1)}{x^3+1}=1+\frac{x-1}{(x+1)(x^2-x+1)}$

Letting $\displaystyle\frac{x-1}{(x+1)(x^2-x+1)}=\frac{P}{x+1}+\frac{Qx+R}{x^2-x+1}$,

Determining the values of P, Q, and R,

$$x-1=P(x^2-x+1)+(Qx+R)(x+1) \quad \text{—①}$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P+Q=0 \\ -P+Q+R=1 \\ P+R=-1 \end{cases} \qquad \therefore P=-\frac{2}{3},\ Q=\frac{2}{3},\ R=-\frac{1}{3}$$

$$\therefore \int \frac{x(x^2+1)}{x^3+1}dx=\int\left[1-\frac{2}{3(x+1)}+\frac{2x-1}{3(x^2-x+1)}\right]dx$$

$$=x-\frac{2}{3}\ln|x+1|+\frac{1}{3}\ln|x^2-x+1|+C \quad \left[=x+\frac{1}{3}\ln\frac{x^2-x+1}{(x+1)^2}+C\right]$$

(2) $\displaystyle\int \frac{x^3-3x-4}{x^3-1}dx$

[Sol] Dividing, $\displaystyle\frac{x^3-3x-4}{x^3-1}=1-\frac{3x+3}{(x-1)(x^2+x+1)}$

Letting $\displaystyle\frac{3x+3}{(x-1)(x^2+x+1)}=\frac{P}{x-1}+\frac{Qx+R}{x^2+x+1}$,

Determining the values of P, Q, and R,

$$3x+3=P(x^2+x+1)+(Qx+R)(x-1) \quad \text{—①}$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P+Q=0 \\ P-Q+R=3 \\ P-R=3 \end{cases} \qquad \therefore P=2,\ Q=-2,\ R=-1$$

$$\therefore \int \frac{x^3-3x-4}{x^3-1}dx=\int\left(1-\frac{2}{x-1}+\frac{2x+1}{x^2+x+1}\right)dx$$

$$=x-2\ln|x-1|+\ln|x^2+x+1|+C \quad \left[=x+\ln\frac{x^2+x+1}{(x-1)^2}+C\right]$$

O 124 b

(3) $\displaystyle\int \frac{2x^2}{(x^2+1)(x-1)^2}\,dx$

[Sol] Letting $\dfrac{2x^2}{(x^2+1)(x-1)^2} = \dfrac{P}{x-1} + \dfrac{Q}{(x-1)^2} + \dfrac{Rx+S}{x^2+1}$,

Determining the values of P, Q, R, and S,

$$2x^2 = P(x-1)(x^2+1) + Q(x^2+1) + (Rx+S)(x-1)^2 \quad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P+R=0 \\ -P+Q-2R+S=2 \\ P+R-2S=0 \\ -P+Q+S=0 \end{cases} \qquad \therefore P=Q=1,\quad R=-1,\quad S=0$$

$$\therefore \int \frac{2x^2}{(x^2+1)(x-1)^2}\,dx = \int\left[\frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{x}{x^2+1}\right]dx$$

$$= \ln|x-1| - \frac{1}{x-1} - \frac{1}{2}\ln|x^2+1| + C$$

$$\left[= \frac{1}{2}\ln\frac{(x-1)^2}{x^2+1} - \frac{1}{x-1} + C \right]$$

(4) $\displaystyle\int \frac{2x^3+3x}{x^4+2x^2+1}\,dx$

[Sol] Letting $\dfrac{2x^3+3x}{(x^2+1)^2} = \dfrac{Px+Q}{x^2+1} + \dfrac{Rx+S}{(x^2+1)^2}$,

Determining the values of P, Q, R, and S,

$$2x^3+3x = (Px+Q)(x^2+1) + (Rx+S) \quad \cdots ①$$

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P=2 \\ Q=0 \\ P+R=3 \\ Q+S=0 \end{cases} \qquad \therefore P=2,\quad R=1,\quad Q=S=0$$

$$\therefore \int \frac{2x^3+3x}{x^4+2x^2+1}\,dx = \int\left[\frac{2x}{x^2+1} + \frac{x}{(x^2+1)^2}\right]dx$$

$$= \ln(x^2+1) - \frac{1}{2(x^2+1)} + C$$

O 125 a Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Evaluate each of the following integrals.

(1) $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$

[Sol] Letting $x = \sin\theta$, $dx = \cos\theta\, d\theta$ $\left(-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}\right)$

When $x = 0$, $\theta = 0$

When $x = \dfrac{1}{2}$, $\theta = \dfrac{\pi}{6}$

Substituting: $\displaystyle\int_0^{\frac{\pi}{6}} \left[\left(\sqrt{1-\boxed{\sin^2\theta}}\right)\cos\theta\right] d\theta = \int_0^{\frac{\pi}{6}} \left[\left(\sqrt{\boxed{\cos^2\theta}}\right)\cos\theta\right] d\theta$

$$= \int_0^{\frac{\pi}{6}} \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

(2) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4-x^2}\,dx$

[Sol] Letting $x = 2\cdot\boxed{\sin\theta}$, $dx = \boxed{2\cos\theta\, d\theta}$ $\left(-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}\right)$

When $x = -\sqrt{2}$, $\theta = -\dfrac{\pi}{4}$

When $x = \sqrt{2}$, $\theta = \dfrac{\pi}{4}$

Substituting: $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\left(\sqrt{4-4\sin^2\theta}\right)2\cos\theta\right] d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\cos^2\theta\, d\theta$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 2\theta)\, d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 2\left[\left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(-\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right)\right]$$

$$= \pi + 2$$

O 125 b

(3) $\displaystyle\int_0^1 (1+x)\sqrt{1-x^2}\,dx$

[Sol] Letting $x=\sin\theta$, $\quad dx=\cos\theta\,d\theta \quad \left(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right)$

When $x=0$, $\quad \theta=0$

When $x=1$, $\quad \theta=\frac{\pi}{2}$

Substituting: $\displaystyle\int_0^{\frac{\pi}{2}}\left[(1+\sin\theta)\left(\sqrt{1-\sin^2\theta}\right)\cos\theta\right]d\theta=\int_0^{\frac{\pi}{2}}\left[(1+\sin\theta)\cos^2\theta\right]d\theta$

$$=\int_0^{\frac{\pi}{2}}(\cos^2\theta+\sin\theta\cos^2\theta)\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}(1+\cos 2\theta)+\sin\theta\cos^2\theta\right]d\theta$$

$$=\left[\frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)-\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{1}{2}\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right)-\frac{1}{3}\cos^3\frac{\pi}{2}\right]-\left[\frac{1}{2}\left(0+\frac{1}{2}\sin 0\right)-\frac{1}{3}\cos^3 0\right]$$

$$=\frac{\pi}{4}+\frac{1}{3}$$

(4) $\displaystyle\int_0^1\sqrt{\frac{1-x}{1+x}}\,dx$

[Sol] Letting $x=\sin\theta$, $\quad dx=\cos\theta\,d\theta \quad \left(-\frac{\pi}{2}<\theta\le\frac{\pi}{2}\right)$

When $x=0$, $\quad \theta=0$

When $x=1$, $\quad \theta=\frac{\pi}{2}$

Substituting: $\displaystyle\int_0^{\frac{\pi}{2}}\left[\left(\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\right)\cos\theta\right]d\theta=\int_0^{\frac{\pi}{2}}\left[\left(\sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}\right)\cos\theta\right]d\theta$

$$=\int_0^{\frac{\pi}{2}}\left[\left(\frac{1-\sin\theta}{\cos\theta}\right)\cos\theta\right]d\theta$$

$$=\int_0^{\frac{\pi}{2}}(1-\sin\theta)\,d\theta$$

$$=\Big[\theta+\cos\theta\Big]_0^{\frac{\pi}{2}}$$

$$=\left(\frac{\pi}{2}+\cos\frac{\pi}{2}\right)-(0+\cos 0)$$

$$=\frac{\pi}{2}-1$$

Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Evaluate each of the following integrals.

(1) $\displaystyle\int_0^1 \sqrt{2x - x^2}\,dx$

[Sol] Completing the square, $2x - x^2 = 1 - (x-1)^2$

Letting $x - 1 = \sin\theta$, $\quad dx = \cos\theta\, d\theta \quad \left(-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}\right)$

When $\quad x = 0, \quad \theta = -\dfrac{\pi}{2}$

When $\quad x = 1, \quad \theta = 0$

Substituting: $\displaystyle\int_{-\frac{\pi}{2}}^{0} \left[\left(\sqrt{1-\sin^2\theta}\right)\cos\theta\right]d\theta = \int_{-\frac{\pi}{2}}^{0} \cos^2\theta\, d\theta$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{0} (1 + \cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{0}$$

$$= \frac{1}{2}\left[\left(0 + \frac{1}{2}\sin 0\right) - \left(-\frac{\pi}{2} - \frac{1}{2}\sin\pi\right)\right]$$

$$= \frac{\pi}{4}$$

(2) $\displaystyle\int_0^4 \sqrt{4x - x^2}\,dx$

[Sol] Completing the square, $4x - x^2 = 4 - (x-2)^2$

Letting $x - 2 = 2\sin\theta$, $\quad dx = 2\cos\theta\, d\theta \quad \left(-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}\right)$

When $\quad x = 0, \quad \theta = -\dfrac{\pi}{2}$

When $\quad x = 4, \quad \theta = \dfrac{\pi}{2}$

Substituting: $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(\sqrt{4-4\sin^2\theta}\right)2\cos\theta\right]d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\,d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - \left(-\frac{\pi}{2} - \frac{1}{2}\sin\pi\right)\right]$$

$$= 2\pi$$

O 126 b

(3) $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^2}$

[Sol] Letting $x = \tan\theta$, $dx = \boxed{\dfrac{1}{\cos^2\theta}}\, d\theta \quad \left(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\right)$

When $x = 0$, $\theta = 0$

When $x = 1$, $\theta = \dfrac{\pi}{4}$

Substituting: $$\int_0^1 \frac{d\theta}{(1+\tan^2\theta)^2\cos^2\theta} = \int_0^1 \frac{(\cos^2\theta)^2}{\cos^2\theta}\, d\theta$$

$$= \int_0^1 \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^1$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

(4) $\displaystyle\int \frac{1-x^2}{(1+x^2)^2}\, dx$

[Sol] Letting $x = \tan\theta$, $dx = \dfrac{1}{\cos^2\theta}\, d\theta \quad \left(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\right)$

Substituting: $$\int \frac{1-\tan^2\theta}{(1+\tan^2\theta)^2\cos^2\theta}\, d\theta = \int \frac{(\cos^2\theta)^2\left(\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}\right)}{\cos^2\theta}\, d\theta$$

$$= \int (\cos^2\theta - \sin^2\theta)\, d\theta$$

$$= \int \cos 2\theta\, d\theta$$

$$= \frac{1}{2}\sin 2\theta + C$$

KUMON

O 127 a

Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Evaluate each of the following integrals.

(1) $\displaystyle\int_0^4 \frac{dx}{\sqrt{x^2+9}}$

[Sol] Letting $x=3\tan\theta$, $\quad dx=\dfrac{3}{\cos^2\theta}d\theta \quad \left(-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}\right)$

When $x=0$, $\quad \theta=0$

When $x=4$, $\quad \theta=a$, $\quad$ where $\tan\theta=\dfrac{4}{3}$

Substituting: $\displaystyle\int_0^a \frac{3}{\cos^2\theta\sqrt{9\tan^2\theta+9}}d\theta=\int_0^a \frac{3}{\cos^2\theta\left(\dfrac{3}{\cos\theta}\right)}d\theta$

$$=\int_0^a \frac{d\theta}{\cos\theta}=\int_0^a \frac{\cos\theta}{1-\sin^2\theta}d\theta$$

$$=\int_0^a\left[\frac{1}{2}\left(\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}\right)(\sin\theta)'\right]d\theta$$

$$=\frac{1}{2}\Big[\ln|1+\sin\theta|-\ln|1-\sin\theta|\Big]_0^a$$

$$=\frac{1}{2}\left[\left(\ln\frac{9}{5}-\ln\frac{1}{5}\right)-(\ln 1-\ln 1)\right] \quad \left(\because \tan a=\frac{4}{3}\rightarrow \sin a=\frac{4}{5}\right)$$

$$=\ln 3$$

(2) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{x^2+9}$

[Sol] Letting $x=3\tan\theta$, $\quad dx=\dfrac{3}{\cos^2\theta}d\theta \quad \left(-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}\right)$

When $x=-\sqrt{3}$, $\quad \theta=-\dfrac{\pi}{6}$

When $x=\sqrt{3}$, $\quad \theta=\dfrac{\pi}{6}$

Substituting: $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{3}{\cos^2\theta(9\tan^2\theta+9)}d\theta=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{3}{\cos^2\theta\left(\dfrac{9}{\cos^2\theta}\right)}d\theta$

$$=\frac{1}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\theta$$

$$=\frac{1}{3}\Big[\theta\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$=\frac{\pi}{9}$$

O 127 b

(3) $\displaystyle\int_0^2 \frac{dx}{1-x+x^2}$

[Sol] Completing the square, $1-x+x^2=\left(x-\dfrac{1}{2}\right)^2+\dfrac{3}{4}$

Letting $x-\dfrac{1}{2}=\dfrac{\sqrt{3}}{2}\tan\theta$, $\quad dx=\dfrac{\sqrt{3}}{2\cos^2\theta}d\theta \qquad \left(-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}\right)$

When $\quad x=0, \qquad \theta=-\dfrac{\pi}{6}$

When $\quad x=2, \qquad \theta=\dfrac{\pi}{3}$

Substituting: $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{3}}{2\cos^2\theta\left(\frac{3}{4}\tan^2\theta+\frac{3}{4}\right)}d\theta=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{3}}{\frac{3}{2}\cos^2\theta\left(\frac{1}{\cos^2\theta}\right)}d\theta$

$$=\frac{2\sqrt{3}}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta$$

$$=\frac{2\sqrt{3}}{3}\Big[\theta\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\frac{\sqrt{3}}{3}\pi$$

(4) $\displaystyle\int_0^{\sin\alpha} \frac{dx}{\sqrt{1-x^2}} \qquad (0<\alpha<\pi)$

[Sol] Letting $x=\sin\theta$, $\qquad dx=\cos\theta\, d\theta \qquad \left(-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}\right)$

When $\quad x=0, \qquad \theta=0$

When $\quad x=\sin\alpha, \qquad \theta=\alpha$

Substituting: $\displaystyle\int_0^{\alpha} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta=\int_0^{\alpha} d\theta$

$$=\Big[\theta\Big]_0^{\alpha}$$

$$=\alpha$$

O 128 a Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Evaluate each of the following integrals.

(1) $\displaystyle\int_0^6 \sqrt{|x-3|}\,dx$

[Sol] $|x-3| = \begin{cases} \boxed{3-x} & (0 \leq x \leq 3) \\ \boxed{x-3} & (3 \leq x \leq 6) \end{cases}$

$$\int_0^6 \sqrt{|x-3|}\,dx = \int_0^3 \sqrt{\boxed{3-x}}\,dx + \int_3^6 \sqrt{\boxed{x-3}}\,dx$$

$$= \left[-\frac{2}{3}(3-x)^{\frac{3}{2}}\right]_0^3 + \left[\frac{2}{3}(x-3)^{\frac{3}{2}}\right]_3^6$$

$$= -\frac{2}{3}(0-3^{\frac{3}{2}}) + \frac{2}{3}(3^{\frac{3}{2}}-0)$$

$$= 4\sqrt{3}$$

(2) $\displaystyle\int_0^\pi \left|\sin x - \sqrt{3}\cos x\right| dx$

[Sol] $\sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$

$$\left|2\sin\left(x-\frac{\pi}{3}\right)\right| = \begin{cases} -2\sin\left(x-\frac{\pi}{3}\right) & \left(0 \leq x \leq \frac{\pi}{3}\right) \\ 2\sin\left(x-\frac{\pi}{3}\right) & \left(\frac{\pi}{3} \leq x \leq \pi\right) \end{cases}$$

$$\int_0^\pi \left|\sin x - \sqrt{3}\cos x\right| dx = -2\int_0^{\frac{\pi}{3}} \sin\left(x-\frac{\pi}{3}\right)dx + 2\int_{\frac{\pi}{3}}^{\pi} \sin\left(x-\frac{\pi}{3}\right)dx$$

$$= 2\left[\cos\left(x-\frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}} - 2\left[\cos\left(x-\frac{\pi}{3}\right)\right]_{\frac{\pi}{3}}^{\pi}$$

$$= 2\left(1-\frac{1}{2}\right) - 2\left(-\frac{1}{2}-1\right)$$

$$= 4$$

O 128 b

(3) $\displaystyle\int_0^2 |(3x-1)(x-1)|\,dx$

[Sol] $|(3x-1)(x-1)| = \begin{cases} (3x-1)(x-1) & \left(0 \le x \le \frac{1}{3},\ 1 \le x \le 2\right) \\ -(3x-1)(x-1) & \left(\frac{1}{3} \le x \le 1\right) \end{cases}$

$$\int_0^2 |(3x-1)(x-1)|\,dx$$
$$= \int_0^{\frac{1}{3}} (3x^2-4x+1)\,dx - \int_{\frac{1}{3}}^1 (3x^2-4x+1)\,dx + \int_1^2 (3x^2-4x+1)\,dx$$
$$= \left[x^3-2x^2+x\right]_0^{\frac{1}{3}} - \left[x^3-2x^2+x\right]_{\frac{1}{3}}^1 + \left[x^3-2x^2+x\right]_1^2$$
$$= \left(\frac{1}{27}-\frac{2}{9}+\frac{1}{3}\right) - (1-2+1) + \left(\frac{1}{27}-\frac{2}{9}+\frac{1}{3}\right) + (8-8+2) - (1-2+1)$$
$$= \frac{62}{27}$$

(4) $\displaystyle\int_a^e |x\ln x|\,dx \qquad (0<a<e)$ (**Hint:** Consider the different cases of a.)

[Sol] $|x\ln x| = \begin{cases} x\ln x & (1 \le x \le e) \\ -x\ln x & (0 < x \le 1) \end{cases}$

When $a \le 1$,

$$\int_a^1 (-x\ln x)\,dx + \int_1^e (x\ln x)\,dx = -\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_a^1 + \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_1^e$$
$$= -\left[-\frac{1}{4} - \left(\frac{a^2}{2}\ln a - \frac{a^2}{4}\right)\right] + \left[\frac{e^2}{2} - \frac{e^2}{4} - \left(-\frac{1}{4}\right)\right]$$
$$= \frac{e^2}{4} + \frac{a^2}{4}(2\ln a - 1) + \frac{1}{2}$$

When $a > 1$,

$$\int_a^e (x\ln x)\,dx = \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_a^e$$
$$= \left(\frac{e^2}{2} - \frac{e^2}{4}\right) - \left(\frac{a^2}{2}\ln a - \frac{a^2}{4}\right)$$
$$= \frac{e^2}{4} - \frac{a^2}{4}(2\ln a - 1)$$

O 129 a Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

1. Determine the value of the constant a at which the integral $\int_0^{\frac{\pi}{2}} |\cos x - \cos a|\,dx \quad \left(0 \leq a \leq \frac{\pi}{2}\right)$, is at a minimum. Then find the minimum value.

[Sol] $\int_0^{\frac{\pi}{2}} |\cos x - \cos a|\,dx = \int_0^{\boxed{a}} (\cos x - \cos a)\,dx + \int_{\boxed{a}}^{\frac{\pi}{2}} \boxed{(\cos a - \cos x)}\,dx$

$$= \Big[\sin x - x\cos a\Big]_0^{\boxed{a}} + \boxed{\Big[x\cos a - \sin x\Big]_a^{\frac{\pi}{2}}}$$

$$= (\sin a - a\cos a) - 0 + \left(\frac{\pi}{2}\cos a - 1\right) - (a\cos a - \sin a)$$

$$= 2\sin a - \left(2a - \frac{\pi}{2}\right)\cos a - 1$$

Letting $f(a) = 2\sin a - \left(2a - \frac{\pi}{2}\right)\cos a - 1$,

$$f'(a) = 2\cos a - \left[2\cos a + \left(2a - \frac{\pi}{2}\right)(-\sin a)\right]$$

$$= \left(2a - \frac{\pi}{2}\right)\sin a$$

a	0	$\cdots$	$\frac{\pi}{4}$	$\cdots$	$\frac{\pi}{2}$
$f'(a)$	0	$-$	0	$+$	$+$
$f(a)$	$\frac{\pi}{2} - 1$	$\searrow$	min.	$\nearrow$	1

$f(a)$ reaches a minimum at $a = \frac{\pi}{4}$.

When $a = \frac{\pi}{4}$, the minimum value is: $\sqrt{2} - 1$

O 129 b

2. Determine the value of a_n, where $a_n = \int_0^{2\pi} (|\sin t| \cos nt)\, dt$.

(*n* is a natural number.)

(**Hint:** Consider the cases when $n = 1$ and $n \geq 2$.)

[Sol]

(a) When $n = 1$,

$$a_n = \int_0^{2\pi} (|\sin t| \cos t)\, dt = \int_0^{\pi} (\sin t \cos t)\, dt - \int_{\pi}^{2\pi} (\sin t \cos t)\, dt$$

$$= \frac{1}{2}\int_0^{\pi} \sin 2t\, dt - \frac{1}{2}\int_{\pi}^{2\pi} \sin 2t\, dt$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2t\right]_0^{\pi} - \frac{1}{2}\left[-\frac{1}{2}\cos 2t\right]_{\pi}^{2\pi}$$

$$= 0$$

(b) When $n \geq 2$,

$$a_n = \int_0^{\pi} (\sin t \cos nt)\, dt - \int_{\pi}^{2\pi} (\sin t \cos nt)\, dt$$

$$= \int_0^{\pi} \frac{1}{2}\{\sin[(1+n)t] + \sin[(1-n)t]\}\, dt - \int_{\pi}^{2\pi} \frac{1}{2}\{\sin[(1+n)t] + \sin[(1-n)t]\}\, dt$$

$$= \frac{1}{2}\left[-\frac{\cos[(n+1)t]}{(n+1)} - \frac{\cos[(1-n)t]}{(1-n)}\right]_0^{\pi} - \frac{1}{2}\left[-\frac{\cos[(n+1)t]}{(n+1)} - \frac{\cos[(1-n)t]}{(1-n)}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{n+1} - \frac{1}{n-1} + \frac{\cos[(n-1)\pi]}{n-1} - \frac{\cos[(n+1)\pi]}{n+1}$$

When n is even, $a_n = \frac{1}{n+1} - \frac{1}{n-1} - \frac{1}{n-1} + \frac{1}{n+1} = -\frac{4}{n^2-1}$

When n is odd, $a_n = \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n+1} = 0$

$n = 1$ also holds true for (b).

Therefore, for all natural numbers, n,

$$\begin{cases} \text{When } n \text{ is even}, & a_n = -\dfrac{4}{n^2-1} \\ \text{When } n \text{ is odd}, & a_n = 0 \end{cases}$$

O 130 a

Advanced Integration

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Evaluate each of the following integrals.

(1) $\displaystyle\int \frac{x^5+2x^4-8x^3-16x^2+x-18}{x^4-8x^2-9}\,dx$

[Sol] $\dfrac{x^5+2x^4-8x^3-16x^2+x-18}{x^4-8x^2-9}=x+2+\dfrac{10x}{(x^2-9)(x^2+1)}$

Letting $\dfrac{10x}{(x^2-9)(x^2+1)}=\dfrac{P}{x+3}+\dfrac{Q}{x-3}+\dfrac{Rx+S}{x^2+1}$,

Determining the values of P, Q, R, and S.

$10x=P(x-3)(x^2+1)+Q(x+3)(x^2+1)+(Rx+S)(x+3)(x-3)$ …①

Matching the coefficients of the left- and right-hand sides of equation ①,

$$\begin{cases} P+Q+R=0 \\ -3P+3Q+S=0 \\ P+Q-9R=10 \\ -3P+3Q-9S=0 \end{cases} \qquad \therefore P=Q=\frac{1}{2},\quad R=-1,\quad S=0$$

$$\therefore \int\frac{x^5+2x^4-8x^3-16x^2+x-18}{x^4-8x^2-9}\,dx=\int\left[x+2+\frac{1}{2(x+3)}+\frac{1}{2(x-3)}-\frac{x}{x^2+1}\right]dx$$

$$=\frac{1}{2}x^2+2x+\frac{1}{2}\ln|x+3|+\frac{1}{2}\ln|x-3|-\frac{1}{2}\ln(x^2+1)+C$$

$$\left[=\frac{1}{2}x^2+2x+\frac{1}{2}\ln\frac{|(x+3)(x-3)|}{x^2+1}+C\right]$$

(2) $\displaystyle\int_{-1}^{\sqrt{3}}\frac{dx}{\sqrt{4-x^2}}$

[Sol] Letting $x=2\sin\theta$, $dx=2\cos\theta\,d\theta$ $\left(-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}\right)$

When $x=-1$, $\theta=-\dfrac{\pi}{6}$

When $x=\sqrt{3}$, $\theta=\dfrac{\pi}{3}$

Substituting: $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{2\cos\theta}{2\cos\theta}\,d\theta$

$$=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}d\theta$$

$$=\Big[\theta\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\frac{\pi}{2}$$

O 130 b

2. Evaluate the following integral.

$$\int_0^{\frac{\pi}{2}} |a\cos x - \sin 2x|\,dx \qquad (a>0)$$

(**Hint:** Consider the cases when $0<a\leq 2$ and $a>2$.)

[Sol] $f(x) = a\cos x - 2\sin x\cos x = 2(\cos x)\left(\frac{a}{2} - \sin x\right)$

(a) When $0<a\leq 2$, $\sin x = \frac{a}{2}$ holds true for one value of x only.

Letting α be that value, $\sin\alpha = \frac{a}{2}$

$$\begin{aligned}
\int_0^{\alpha}(a\cos x - \sin 2x)dx + \int_{\alpha}^{\frac{\pi}{2}}(\sin 2x - a\cos x)dx &= \left[a\sin x + \frac{1}{2}\cos 2x\right]_0^{\alpha} + \left[-\frac{1}{2}\cos 2x - a\sin x\right]_{\alpha}^{\frac{\pi}{2}} \\
&= \left(a\sin\alpha + \frac{1}{2}\cos 2\alpha - \frac{1}{2}\right) + \left(\frac{1}{2} - a + \frac{1}{2}\cos 2\alpha + a\sin\alpha\right) \\
&= 2a\sin\alpha + \cos 2\alpha - a \\
&= 2a\sin\alpha + 1 - 2\sin^2\alpha - a \\
&= 2a\left(\frac{a}{2}\right) + 1 - 2\left(\frac{a}{2}\right)^2 - a \qquad \left(\because \sin\alpha = \frac{a}{2}\right) \\
&= 1 - a + \frac{a^2}{2}
\end{aligned}$$

(b) When $a>2$, then $\frac{a}{2} - \sin x > 0$

Therefore,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}(a\cos x - \sin 2x)dx &= \left[a\sin x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} \\
&= a - \frac{1}{2} - \frac{1}{2} \\
&= a - 1
\end{aligned}$$

Thus, $\int_0^{\frac{\pi}{2}} |a\cos x - \sin 2x|\,dx = \begin{cases} 1 - a + \frac{a^2}{2} & (\text{when } 0<a\leq 2) \\ a - 1 & (\text{when } a>2) \end{cases}$

O 131 a Applications of Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Given a curve $y=f(x)$, where $a \leq x \leq b$,
Letting S be the area bounded by the curve, the x-axis and the two lines $x=a$ and $x=b$,

$$S=\int_a^b |f(x)|\,dx$$

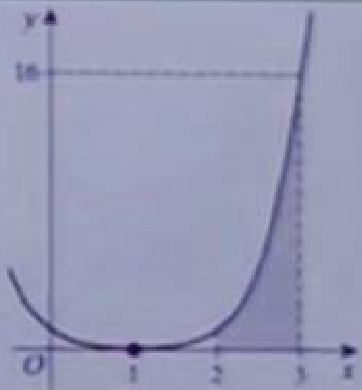
Determine the area bounded by the x-axis and the following curves.

Ex. $y=(x-1)^4 \qquad (2 \leq x \leq 3)$

[Sol] When $2 \leq x \leq 3$, $y>0$.

Thus,

$$S=\int_2^3 (x-1)^4\,dx=\frac{1}{5}\Big[(x-1)^5\Big]_2^3$$
$$=\frac{31}{5}$$



(1) $y=(2x+5)^2 \qquad (-3 \leq x \leq -2)$

[Sol] When $-3 \leq x \leq -2$, $y \geq 0$.

Thus,

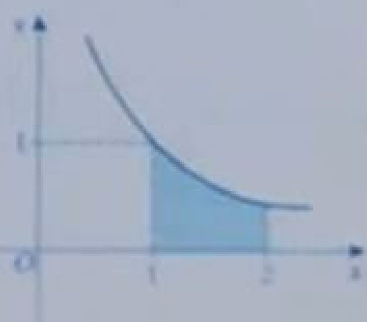
$$S=\int_{-3}^{-2} (2x+5)^2\,dx=\frac{1}{6}\Big[(2x+5)^3\Big]_{-3}^{-2}$$
$$=\frac{1}{3}$$

(2) $y=\dfrac{1}{\sqrt{x}} \qquad (1 \leq x \leq 2)$

[Sol] When $1 \leq x \leq 2$, $y>0$.

Thus,

$$S=\int_1^2 \frac{1}{\sqrt{x}}\,dx=\Big[2\sqrt{x}\Big]_1^2$$
$$=2(\sqrt{2}-1)$$



O 131 b

Ex. $y=x^3-x^2-x-1 \qquad (0\leq x\leq 1)$

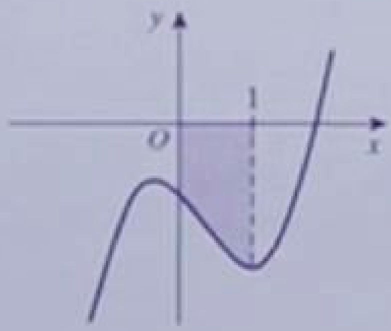
[Sol] When $0\leq x\leq 1$, $y<0$.

Thus,

$$S=-\int_0^1 (x^3-x^2-x-1)\,dx$$

$$=-\left[\frac{x^4}{4}-\frac{x^3}{3}-\frac{x^2}{2}-x\right]_0^1$$

$$=\frac{19}{12}$$



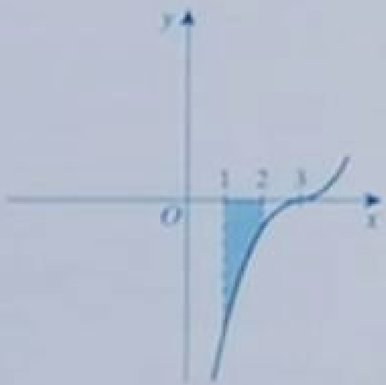
(3) $y=(x-3)^3 \qquad (1\leq x\leq 2)$

[Sol] When $1\leq x\leq 2$, $y<0$.

Thus,

$$S=-\int_1^2 (x-3)^3\,dx=-\frac{1}{4}\left[(x-3)^4\right]_1^2$$

$$=\frac{15}{4}$$



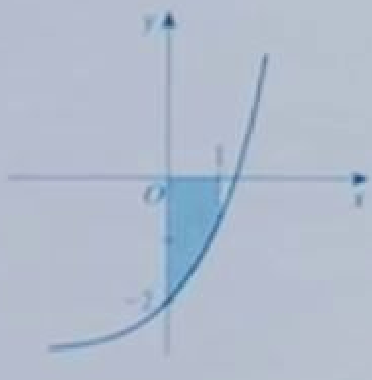
(4) $y=e^x-3 \qquad (0\leq x\leq 1)$

[Sol] When $0\leq x\leq 1$, $y<0$.

Thus,

$$S=-\int_0^1 (e^x-3)\,dx=-\left[e^x-3x\right]_0^1$$

$$=4-e$$



O 132 a Applications of Integrals 1

Time : to : Date Name

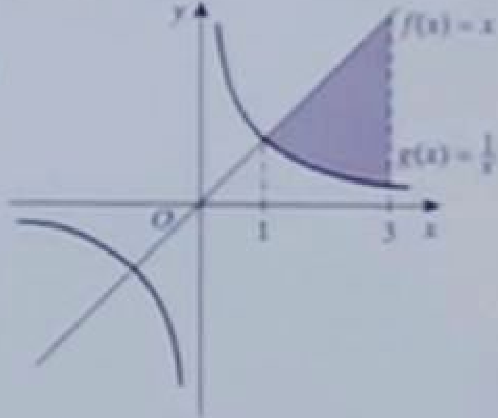
100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

Given two curves $y = f(x)$ and $y = g(x)$, where $a \leq x \leq b$,
Letting S be the area bounded by the two curves and the two lines $x = a$ and $x = b$,

$$S = \int_a^b |f(x) - g(x)|\, dx$$

1. Determine the area bounded by the following curves.

(1) $f(x) = x, \quad g(x) = \dfrac{1}{x} \qquad (1 \leq x \leq 3)$



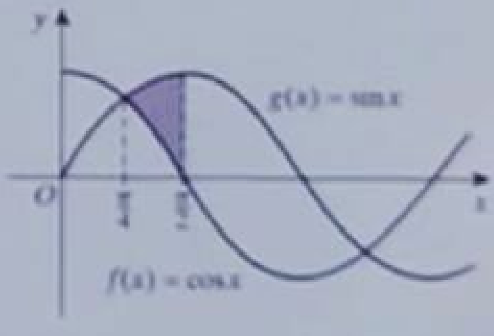
[Sol] When $1 \leq x \leq 3$, $f(x) > 0$ and $g(x) > 0$

Also, $x \geq \dfrac{1}{x}$ over $1 \leq x \leq 3$

Thus,

$$S = \int_1^3 \left(x - \frac{1}{x}\right) dx = \left[\frac{1}{2}x^2 - \ln x\right]_1^3$$
$$= 4 - \ln 3$$

(2) $f(x) = \cos x, \quad g(x) = \sin x \qquad \left(\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{2}\right)$



[Sol] When $\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{2}$, $f(x) \geq 0$ and $g(x) > 0$

Also, $\cos x \leq \sin x$ over $\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{2}$

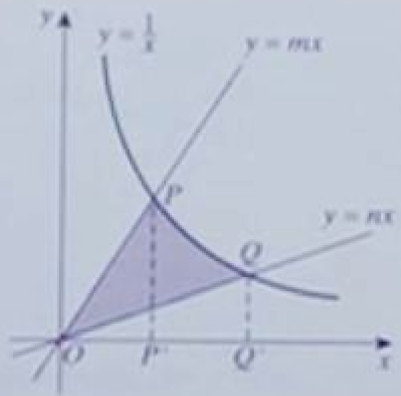
Thus,

$$S = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\, dx = \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= -1 + \sqrt{2}$$

O 132 b

2. Determine the area bounded by the curve $y=\dfrac{1}{x}$ and the two lines $y=mx$ and $y=nx$ (where $m>n>0$) in the 1st Quadrant.

[Sol] Given the diagram on the right, let p and q be the x-coordinates of points P' and Q', respectively, and let S be the area of the shaded region.



$$S=\triangle OPP'+\int_p^q \frac{1}{x}\,dx-\triangle OQQ' \quad \ldots①$$

Finding the points of intersection,

The coordinates of P and Q are: $\left(\dfrac{1}{\sqrt{m}}, \boxed{\sqrt{m}}\right), \left(\boxed{\dfrac{1}{\sqrt{n}}}, \boxed{\sqrt{n}}\right)$.

Thus, from ①,

$$S=\frac{1}{2}\cdot\frac{1}{\sqrt{m}}\cdot\sqrt{m}+\int_{\frac{1}{\sqrt{m}}}^{\frac{1}{\sqrt{n}}}\frac{1}{x}\,dx-\frac{1}{2}\cdot\frac{1}{\sqrt{n}}\cdot\sqrt{n}$$

$$=\Big[\ln x\Big]_{\frac{1}{\sqrt{m}}}^{\frac{1}{\sqrt{n}}}$$

$$=\ln\frac{1}{\sqrt{n}}-\ln\frac{1}{\sqrt{m}}$$

$$=\ln\sqrt{\frac{m}{n}}$$

$$\left[=\frac{1}{2}\ln\frac{m}{n}\right]$$

KUMON

O 133 a Applications of Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Determine the area of a circle with radius r and center at the origin.

[Sol] The equation of a circle is $x^2+y^2=r^2$.

$$\therefore\ y=\pm\sqrt{r^2-x^2}$$

Letting S be the area of the circle,

$$S=\int_{-r}^{r}\sqrt{r^2-x^2}\,dx-\int_{-r}^{r}\left(-\sqrt{r^2-x^2}\right)dx$$

$$=\int_{-r}^{r}\left[\sqrt{r^2-x^2}-\left(-\sqrt{r^2-x^2}\right)\right]dx$$

$$=2\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

Letting $x=r\sin\theta\ \left(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$, $dx=r\cos\theta\,d\theta$

When $x=-r,\ \theta=-\frac{\pi}{2}$; when $x=r,\ \theta=\frac{\pi}{2}$

In this domain, $\cos\theta\geq 0$

Therefore, $\sqrt{r^2-x^2}=\sqrt{r^2\cos^2\theta}=|r\cos\theta|=r\cos\theta$

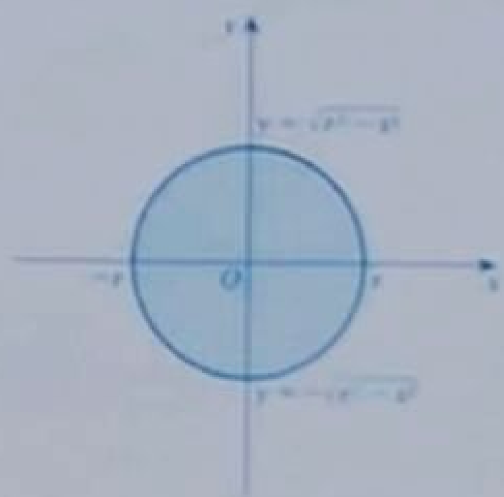
$$\therefore\ S=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}r\cos\theta\cdot r\cos\theta\,d\theta$$

$$=2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$=r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$

$$=r^2\left[\theta+\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\pi r^2$$



O 133 b

2. Determine the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (where $a>0, b>0$)

[Sol] $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

Letting S be the area of the ellipse,

$$S=2\int_{-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx$$

Letting $x=a\sin\theta$ $\left(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$, $dx=a\cos\theta\,d\theta$

When $x=-a$, $\theta=-\frac{\pi}{2}$; when $x=a$, $\theta=\frac{\pi}{2}$

Therefore,

$$S=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{b}{a}\cdot a\cos\theta\cdot a\cos\theta\,d\theta$$

$$=2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=ab\left[\theta+\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\pi ab$$

KUMON
O 134 a

Applications of Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Draw a rough graph of the curve $y = \sin x - \cos x$ $(0 \le x \le 2\pi)$, and then determine the area above the x-axis bounded by the curve and the x-axis.

[Sol] Finding the x-axis intercepts.

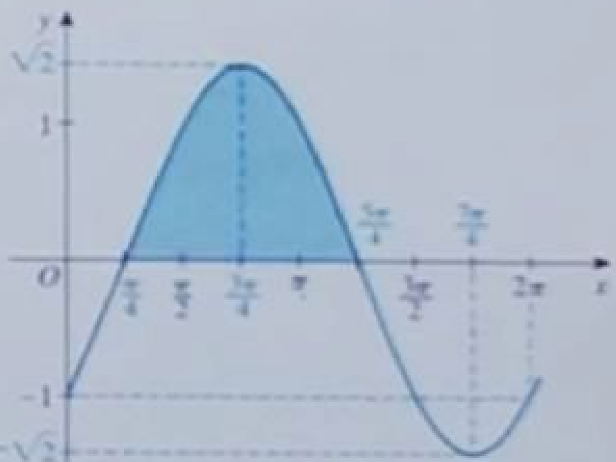
$$y = \sin x - \cos x = 0$$

$$\sin x = \cos x$$

$$\tan x = 1$$

$$\therefore\ y = 0 \text{ when } x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$y' = \cos x + \sin x$$



x	0	$\cdots$	$\frac{3\pi}{4}$	$\cdots$	$\frac{7\pi}{4}$	$\cdots$	2π
y'	+	+	0	−	0	+	+
y	-1	↗	$\sqrt{2}$	↘	$-\sqrt{2}$	↗	-1

Letting S be the area,

$$S = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$$

$$= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= 2\sqrt{2}$$

O 134 b

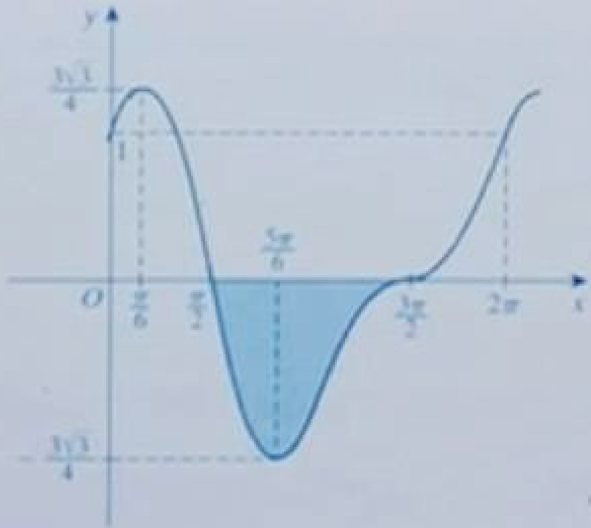
2. Draw a rough graph of the curve $y = \cos x(1 + \sin x)$ $(0 \le x \le 2\pi)$, and then determine the area below the x-axis bounded by the curve and the x-axis.

[Sol] Finding the x-axis intercepts,

$$y = \cos x(1 + \sin x) = 0$$

$$\therefore \cos x = 0 \text{ or } \sin x = -1$$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$



$$y = \cos x(1 + \sin x)$$

$$= \cos x + \frac{1}{2}\sin 2x$$

$$y' = -\sin x + \cos 2x$$

$$= -\sin x + 1 - 2\sin^2 x$$

$$= -2\sin^2 x - \sin x + 1$$

$$= -(2\sin x - 1)(\sin x + 1)$$

x	0	$\cdots$	$\frac{\pi}{6}$	$\cdots$	$\frac{5\pi}{6}$	$\cdots$	$\frac{3\pi}{2}$	$\cdots$	2π
y'	+	+	0	−	0	+	0	+	+
y	1	↗	$\frac{3\sqrt{3}}{4}$	↘	$-\frac{3\sqrt{3}}{4}$	↗	0	↗	1

Letting S be the area,

$$S = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\cos x + \frac{1}{2}\sin 2x\right) dx$$

$$= -\left[\sin x - \frac{1}{4}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= 2$$

Applications of Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Determine the area bounded by the curves $y=\frac{1}{2}x+\sqrt{x}$ and $y=x$. $(x\geqq 0)$

[Sol] Finding the points of intersection.

$$\frac{1}{2}x+\sqrt{x}=x$$

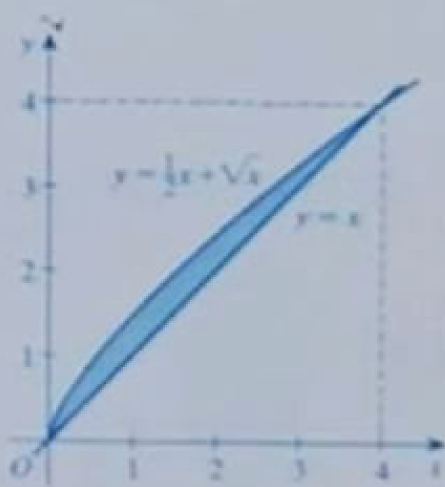
$$\frac{1}{2}x-\sqrt{x}=0$$

$$\frac{1}{2}\sqrt{x}\left(\sqrt{x}-2\right)=0$$

$$\therefore\ x=0,\ 4$$

Thus, the graphs intersect at $(0, 0)$ and at $(4, 4)$.

The graph is shown at right.



Letting S be the area,

$$S=\int_0^4\left(\frac{1}{2}x+\sqrt{x}-x\right)dx$$

$$=\int_0^4\left(\sqrt{x}-\frac{1}{2}x\right)dx$$

$$=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{4}x^2\right]_0^4$$

$$=\frac{16}{3}-4$$

$$=\frac{4}{3}$$

O 135 b

2. Determine the area bounded by the curves $y = \sin x$ and $y = \sin\left(x - \frac{\pi}{3}\right)$.
$(0 \leq x \leq 2\pi)$

[Sol] Finding the points of intersection,

$$\sin x = \sin\left(x - \frac{\pi}{3}\right)$$

$$\sin x - \sin\left(x - \frac{\pi}{3}\right) = 0$$

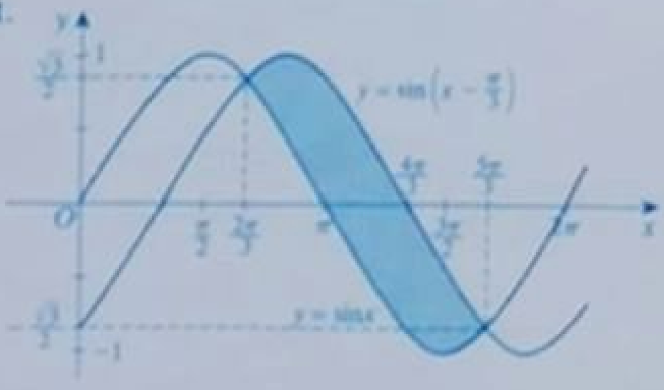
$$2\cos\left(\frac{x + x - \frac{\pi}{3}}{2}\right)\sin\left(\frac{x - x + \frac{\pi}{3}}{2}\right) = 0$$

$$\therefore \cos\left(x - \frac{\pi}{6}\right) = 0$$

$$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3}$$

Thus, the graphs intersect at $\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and at $\left(\frac{5\pi}{3}, -\frac{\sqrt{3}}{2}\right)$.

The graph is shown at right.



Letting S be the area,

$$S = \int_{\frac{2\pi}{3}}^{\frac{5\pi}{3}} \left[\sin\left(x - \frac{\pi}{3}\right) - \sin x\right] dx$$

$$= \left[-\cos\left(x - \frac{\pi}{3}\right) + \cos x\right]_{\frac{2\pi}{3}}^{\frac{5\pi}{3}}$$

$$= 2$$

O 136 a Applications of Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

1. Determine the area bounded by the curves $y=\sqrt{3}\cos 2x$ and $y=\sin x$. $(0<x<\pi)$

(**Hint:** It is not necessary to find the exact values of the points of intersection.)

[Sol] Finding the points of intersection,

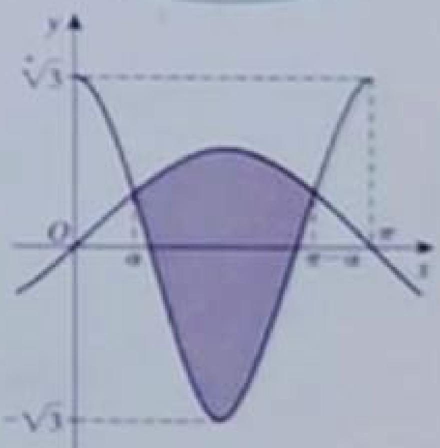
$$\sqrt{3}\cos 2x=\sin x$$
$$\sqrt{3}\left(\boxed{1-2\sin^2 x}\right)=\sin x$$
$$(2\sin x+\sqrt{3})(\sqrt{3}\sin x-\boxed{1})=0$$
$$\sin x=\boxed{\frac{\sqrt{3}}{3}} \qquad (\because \text{ when } 0<x<\pi,\ 2\sin x+\sqrt{3}\neq 0)$$
$$\therefore \sin\alpha=\frac{\sqrt{3}}{3}, \quad \text{where } 0<\alpha<\frac{\pi}{2} \quad \text{—①}$$

Thus, the points of intersection are at:

$$x=\alpha \text{ and } x=\pi-\alpha$$

Remember: $\sin\alpha=\sin(\pi-\alpha)$

The graph is shown at right.



Letting S be the area,

$$S=\int_{\alpha}^{\pi-\alpha}(\sin x-\sqrt{3}\cos 2x)\,dx$$
$$=\left[-\cos x-\frac{\sqrt{3}}{2}\sin 2x\right]_{\alpha}^{\pi-\alpha}$$
$$=2\cos\alpha+\sqrt{3}\sin 2\alpha$$

From ①, $\cos\alpha=\dfrac{\sqrt{6}}{3}$

Thus, $S=2\cos\alpha+\sqrt{3}\cdot 2\sin\alpha\cos\alpha$

$$=\frac{2\sqrt{6}}{3}+\frac{2\sqrt{6}}{3}=\frac{4\sqrt{6}}{3}$$

O 136 b

2. Determine the area bounded by the curves $y = -7\cos x$ and $y = 3\cos 2x$.
$\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

[Sol] Finding the points of intersection,

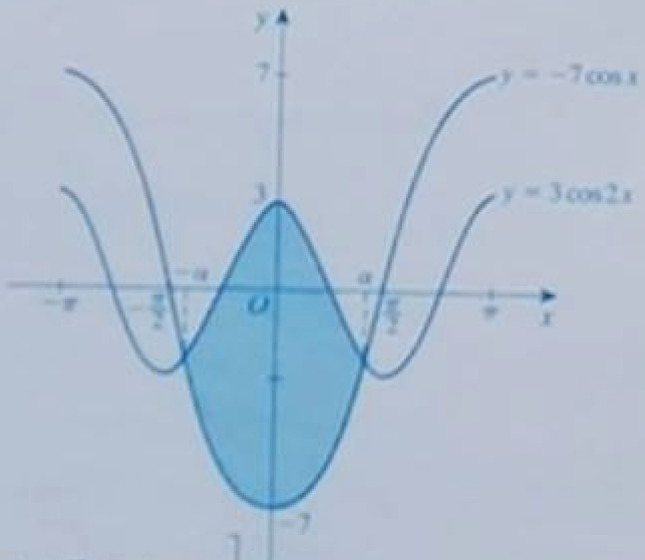
$$-7\cos x = 3\cos 2x$$
$$6\cos^2 x + 7\cos x - 3 = 0$$
$$(3\cos x - 1)(2\cos x + 3) = 0$$
$$\cos x = \frac{1}{3} \qquad (\because 2\cos x + 3 \neq 0)$$
$$\therefore \cos\alpha = \frac{1}{3}, \quad \text{where } 0 < \alpha < \frac{\pi}{2} \qquad \cdots ①$$

Thus, the points of intersection are at:

$$x = \alpha \text{ and } x = -\alpha$$

Remember: $\cos\alpha = \cos(-\alpha)$

The graph is shown at right.



Letting S be the area,

$$S = \int_{-\alpha}^{\alpha} (3\cos 2x + 7\cos x)\,dx$$
$$= \left[\frac{3}{2}\sin 2x + 7\sin x\right]_{-\alpha}^{\alpha}$$
$$= \left(\frac{3}{2}\sin 2\alpha + 7\sin\alpha\right) - \left[\frac{3}{2}\sin(-2\alpha) + 7\sin(-\alpha)\right]$$

$\sin(-\theta) = -\sin\theta$

$$= \left(\frac{3}{2}\sin 2\alpha + 7\sin\alpha\right) + \left(\frac{3}{2}\sin 2\alpha + 7\sin\alpha\right)$$
$$= 3\sin 2\alpha + 14\sin\alpha$$

From ①, $\sin\alpha = \frac{2\sqrt{2}}{3}$

Thus, $S = 3\cdot 2\sin\alpha\cos\alpha + 14\sin\alpha$

$$= 3\cdot 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right) + 14\left(\frac{2\sqrt{2}}{3}\right) = \frac{32\sqrt{2}}{3}$$

Applications of Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Determine the value of a at which the graphs of $y = \ln x$ and $y = ax + 1$ touch at only one point, and for that value, find the area bounded by the x-axis, the y-axis, and the two curves.

[Sol] Let $f(x) = \ln x$ $\quad\cdots①$

Let $g(x) = ax + 1$ $\quad\cdots②$

Let ① and ② touch at $(t,\ at+1)$ (where $t > 0$).

Letting $f(t) = g(t)$,

$$\ln t = \boxed{at + 1} \quad\cdots③$$

Letting $f'(t) = g'(t)$,

$$\frac{1}{t} = a$$

$$\therefore\ at = 1 \quad\cdots④$$

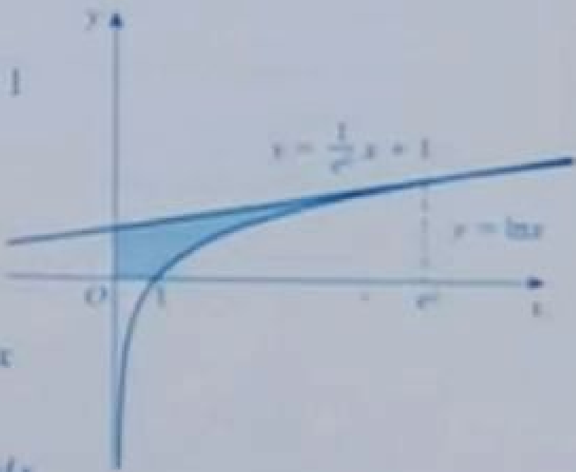
Solving ③ and ④, $\ln t = 2$

$$\therefore\ t = e^2 \quad\cdots⑤$$

Substituting ⑤ into ④,

$$a = \frac{1}{e^2}$$

Thus, $f(x) = \ln x$ and $g(x) = \frac{1}{e^2}x + 1$ touch at only one point, as shown on the right.



Letting S be the area,

$$S = \int_0^{e^2}\left(\frac{1}{e^2}x + 1\right)dx - \int_1^{e^2}\ln x\,dx$$

$$= \frac{1}{e^2}\int_0^{e^2}x\,dx + \int_0^{e^2}dx - \int_1^{e^2}\ln x\,dx$$

$$= \frac{1}{2e^2}\Big[x^2\Big]_0^{e^2} + \Big[x\Big]_0^{e^2} - \Big[x\ln x - x\Big]_1^{e^2}$$

$$= \frac{e^2}{2} - 1$$

O 137 b

2. Determine the value of a at which the graphs of $y = ax^2$ and $y = \ln x$ touch at only one point, and for that value, find the area bounded by the x-axis and the two curves.

[Sol] Let $f(x) = \ln x$ $\quad\cdots①$

Let $g(x) = ax^2$ $\quad\cdots②$

Let ① and ② touch at $(t,\ at^2)$ (where $t > 0$).

Letting $f(t) = g(t)$,

$$\ln t = at^2 \quad\cdots③$$

Letting $f'(t) = g'(t)$,

$$\frac{1}{t} = 2at$$

$$\therefore\ 2at^2 = 1 \quad\cdots④$$

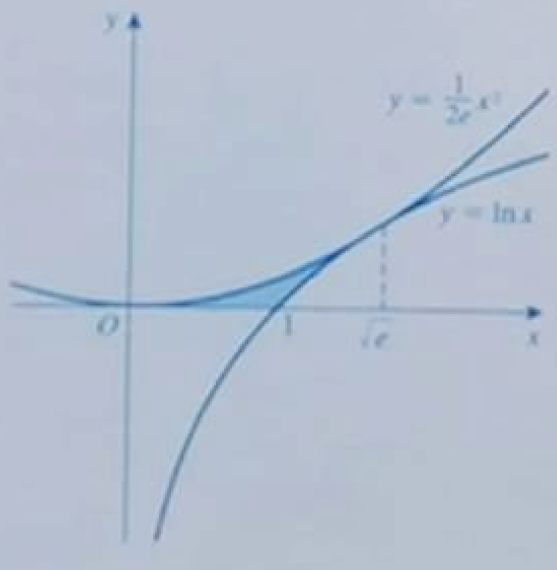
Solving ③ and ④, $\ln t = \dfrac{1}{2}$

$$\therefore\ t = e^{\frac{1}{2}} = \sqrt{e} \quad\cdots⑤$$

Substituting ⑤ into ④,

$$a = \frac{1}{2e}$$

Thus, $f(x) = \ln x$ and $g(x) = \dfrac{1}{2e}x^2$ touch at only one point, as shown on the right.



Letting S be the area,

$$S = \int_0^{\sqrt{e}} \frac{x^2}{2e}\,dx - \int_1^{\sqrt{e}} \ln x\,dx$$

$$= \frac{1}{6e}\Big[x^3\Big]_0^{\sqrt{e}} - \Big[x\ln x - x\Big]_1^{\sqrt{e}}$$

$$= \frac{\sqrt{e}}{6} - \frac{\sqrt{e}}{2} + \sqrt{e} - 1$$

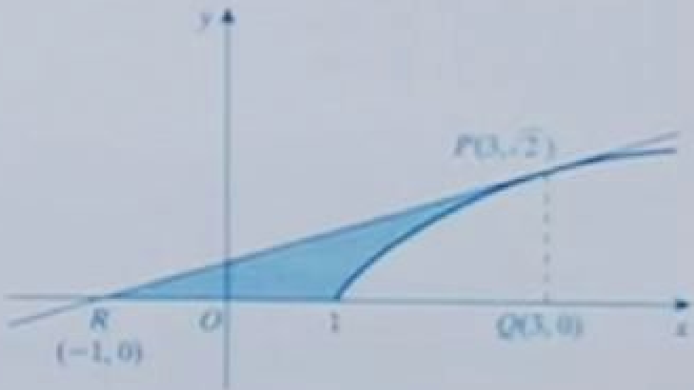
$$= \frac{2\sqrt{e}}{3} - 1$$

O 138 a　　Applications of Integrals 1

Time　:　to　:　　Date　　Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

1. Determine the area bounded by the curve $y=\sqrt{x-1}$, its tangent line at $(3, \sqrt{2})$, and the x-axis.



[Sol] $y=\sqrt{x-1}$　　...①

$$y'=\frac{1}{2\sqrt{x-1}}$$

When $x=3$,

$$y'=\frac{1}{2\sqrt{2}}$$

Finding the equation of the tangent line to ① at $P(3, \sqrt{2})$.

$$y-\sqrt{2}=\frac{1}{2\sqrt{2}}(x-3)$$

$$\therefore\ y=\frac{1}{2\sqrt{2}}x+\frac{1}{2\sqrt{2}}\quad \text{...②}$$

Letting R be the x-intercept of ②.

When $y=0$, $x=-1$.

Therefore, the coordinates of R are: $(-1, 0)$

Letting S be the area.

$$S=\triangle PQR-\int_1^3\sqrt{x-1}\,dx \qquad (\text{where } PQ\perp QR)$$

$$=\frac{1}{2}\cdot 4\cdot\sqrt{2}-\left[\frac{2}{3}\sqrt{(x-1)^3}\right]_1^3$$

$$=2\sqrt{2}-\frac{4\sqrt{2}}{3}$$

$$=\frac{2\sqrt{2}}{3}$$

O 138 b

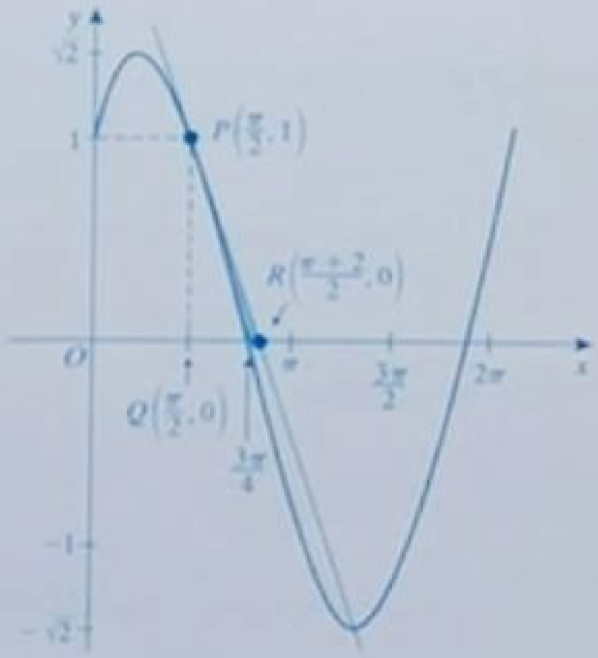
2. Determine the area bounded by the curve $y = \sin x + \cos x$, its tangent line at $\left(\frac{\pi}{2}, 1\right)$, and the x-axis.

[Sol] $y = \sin x + \cos x \quad \cdots ①$

$y' = \cos x - \sin x$

When $x = \frac{\pi}{2}$,

$y' = -1$



Finding the equation of the tangent line to ① at $P\left(\frac{\pi}{2}, 1\right)$,

$$y - 1 = -\left(x - \frac{\pi}{2}\right)$$

$$\therefore\ y = -x + \frac{\pi + 2}{2} \qquad \cdots ②$$

Letting R be the x-intercept of ②,

When $y = 0$, $x = \frac{\pi + 2}{2}$

Therefore, the coordinates of R are: $\left(\frac{\pi + 2}{2}, 0\right)$

Letting S be the area,

$$S = \triangle PQR - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (\sin x + \cos x)\,dx, \quad (\text{where } PQ \perp QR)$$

$$= \frac{1}{2} \cdot 1 \cdot 1 - \Big[-\cos x + \sin x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}$$

$$= \frac{1}{2} - (\sqrt{2} - 1)$$

$$= \frac{3}{2} - \sqrt{2}$$

O 139 a Applications of Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

1. Determine the area bounded by the curve $y = x^3 - 3x^2 + 1$, its tangent line at the point of inflection, and the y-axis.

[Sol] $y' = 3x^2 - 6x$

$= 3x(x-2)$

$y'' = 6x - 6$

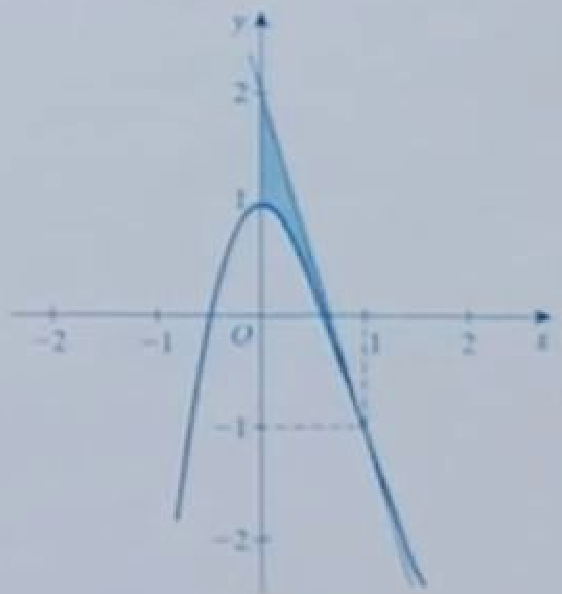
$= 6(x-1)$

Therefore, the point of inflection is: $(\boxed{1}, \boxed{-1})$

Thus, the equation of the tangent line at that point is:

$y + 1 = -3(x-1)$

$\therefore\ y = -3x + 2$



Letting S be the area,

$$S = \int_0^1 \left[(-3x+2) - (x^3 - 3x^2 + 1)\right] dx$$

$$= -\int_0^1 (x^3 - 3x^2 + 3x - 1)\, dx$$

$$= -\left[\frac{x^4}{4} - x^3 + \frac{3x^2}{2} - x\right]_0^1$$

$$= \frac{1}{4}$$

O 139 b

2. Determine the area bounded by the curve $y = xe^{-x}$, its tangent line at the point of inflection, and the y-axis.

[Sol] $y' = e^{-x} - xe^{-x}$

$$= (1-x)e^{-x}$$

$$y'' = -e^{-x} + (1-x)(-e^{-x})$$

$$= (x-2)e^{-x}$$

Therefore, the point of inflection is: $(2, 2e^{-2})$

Thus, the equation of the tangent line at that point is:

$$y - 2e^{-2} = -e^{-2}(x-2)$$

$$\therefore y = -e^{-2}(x-4)$$

Letting S be the area,

$$S = \int_0^2 \left[-e^{-2}(x-4) - xe^{-x} \right] dx$$

$$= -e^{-2}\left[\frac{1}{2}x^2 - 4x\right]_0^2 + \left[xe^{-x}\right]_0^2 - \int_0^2 e^{-x} dx$$

$$= -e^{-2}(2-8) + 2e^{-2} + \left[e^{-x}\right]_0^2$$

$$= 9e^{-2} - 1$$

Applications of Integrals 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Determine the area bounded by the x-axis and the curve $y=\sqrt{x}$. $(1 \leq x \leq 4)$

[Sol] When $1 \leq x \leq 4$, $y>0$.

Thus, $S=\displaystyle\int_1^4 \sqrt{x}\,dx$

$=\dfrac{2}{3}\left[x^{\frac{3}{2}}\right]_1^4=\dfrac{2}{3}(8-1)$

$=\dfrac{14}{3}$

2. Determine the area bounded by the curves $y=e^x$ and $y=x+4$. $(0 \leq x \leq 1)$

[Sol] Letting $f(x)=e^x$ and $g(x)=x+4$.

When $0 \leq x \leq 1$, $f(x)>0$ and $g(x)>0$

Also, $x+4>e^x$ over $0 \leq x \leq 1$

Thus, $S=\displaystyle\int_0^1 (x+4-e^x)\,dx$

$=\left[\dfrac{1}{2}x^2+4x-e^x\right]_0^1=\left(\dfrac{1}{2}+4-e\right)-(-1)$

$=\dfrac{11}{2}-e$

O 140 b

3. Determine the area bounded by the curves $y = \sin x$ and $y = \sin 2x$. $(0 \le x \le \pi)$

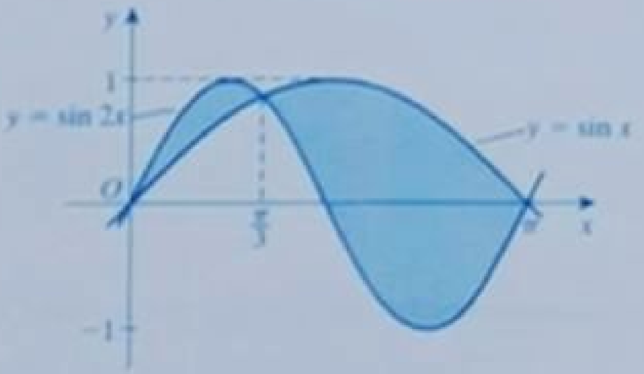
[Sol] Finding the points of intersection.

$$\sin x = \sin 2x$$
$$\sin x(1 - 2\cos x) = 0$$
$$\therefore \ \sin x = 0 \text{ and } \cos x = \frac{1}{2}$$

Over $0 \le x \le \pi$, the x-coordinates of the points of intersection are:

$$x = 0, \frac{\pi}{3}, \pi$$

The graph is shown below.



Letting S be the area,

$$S = \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\, dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin 2x)\, dx$$
$$= \left[-\frac{1}{2}\cos 2x + \cos x \right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2}\cos 2x \right]_{\frac{\pi}{3}}^{\pi}$$
$$= \frac{5}{2}$$

O 141 a Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Letting S be the area bounded by the curves $x = f(t)$ and $y = g(t)$ $(\alpha \leq t \leq \beta)$, the x-axis and the two lines $x = a$ and $x = b$, (where $t \geq 0$)

$$S = \int_a^b |y|\,dx = \int_\alpha^\beta |g(t)f'(t)|\,dt,\ a = f(\alpha),\ b = f(\beta),\ a < b$$

1. Given that for each real value of t, there is a point with coordinates (x, y), where $x = 2t - 5$ and $y = t - 1$, let l be the line formed by the set of these (x, y) points. Determine the area bounded by line l, the x-axis and the two lines $x = -2$ and $x = -1$.

[Sol] When $x = -2$, $t = \dfrac{3}{2}$.

When $x = -1$, $t = \boxed{2}$.

As shown on the graph, each value of t corresponds to a point whose coordinates are (x, y).

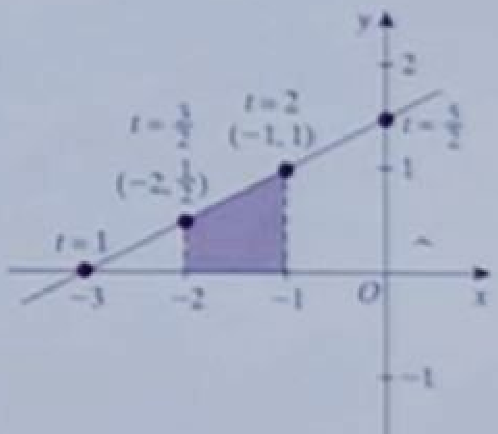
Thus, the curve from $t = \dfrac{3}{2}$ to $t = 2$ and the x-axis bound the shaded area.

$$S = \int_{-2}^{-1} y\,dx$$

$$= \int_{\frac{3}{2}}^{2} y \cdot \frac{dx}{dt} \cdot \boxed{dt}$$

$$= \int_{\frac{3}{2}}^{2} (t-1) \cdot 2 \cdot dt$$

$$= 2\left[\frac{t^2}{2} - t\right]_{\frac{3}{2}}^{2} = \frac{3}{4}$$



O 141 b

2. Given that for each real value of t, there is a point with coordinates (x, y), where $x = t^2 - 4$ and $y = 2t - 1$, let m be the line formed by the set of these (x, y) points. Determine the area bounded by line m, the x-axis and the line $x = -\frac{7}{4}$.

(**Hint:** Find the x-intercept of the curve.)

[Sol] When $y = 0$, $t = \frac{1}{2}$.

When $x = -\frac{7}{4}$, $t = \frac{3}{2}$.

Remember: $t \geq 0$

As shown on the graph, each value of t corresponds to a point whose coordinates are (x, y).

Thus, the curve from $t = \frac{1}{2}$ to $t = \frac{3}{2}$ and the x-axis bound the shaded area.

$$S = \int_{\frac{1}{2}}^{\frac{3}{2}} y \frac{dx}{dt} \, dt$$

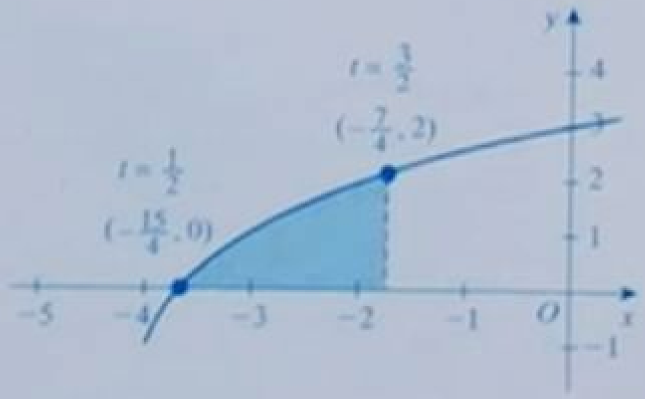
$$= \int_{\frac{1}{2}}^{\frac{3}{2}} (2t - 1)(2t) \, dt$$

$$= 2\int_{\frac{1}{2}}^{\frac{3}{2}} (2t^2 - t) \, dt$$

$$= 2\left[\frac{2}{3}t^3 - \frac{1}{2}t^2\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= 2 \cdot \frac{7}{6}$$

$$= \frac{7}{3}$$



O 142 a Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

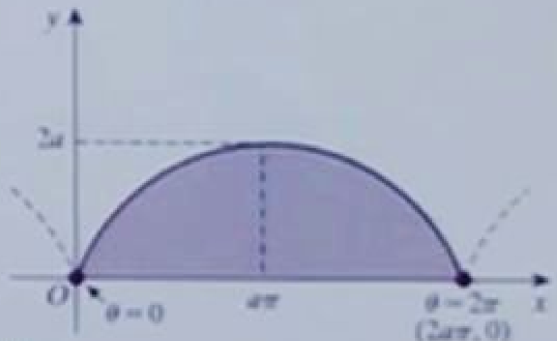
1. Determine the area of the cycloid generated by $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and the x-axis, when $a > 0$ and $0 \leq x \leq 2a\pi$.

[Sol] The curve intersects the x-axis when $a(1 - \cos\theta) = 0$.

$\therefore \theta = \boxed{0}, \boxed{2\pi}$

Thus, the x-intercepts are: $(\boxed{0}, 0)$ and $(\boxed{2a\pi}, 0)$

As shown on the graph, each value of θ corresponds to a point whose coordinates are (x, y).



$$S = \int_0^{2a\pi} y\,dx$$
$$= \int_0^{2\pi} y\frac{dx}{d\theta} \cdot \boxed{d\theta}$$
$$= \int_0^{2\pi} a(1 - \cos\theta) \cdot a(1 - \cos\theta)\,d\theta$$
$$= a^2\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$
$$= a^2\int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right)d\theta$$
$$= a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$
$$= 3\pi a^2$$

Note: A cycloid is the path that a point on the circumference of a circle traces as the circle rolls along a straight line.

O 142 b

2. Determine the area in the 1[st] Quadrant of the figure generated by $x = 2t + \cos t$, $y = \sin t$, when $0 \le x < 2\pi$.

[Sol] The curve intersects the x-axis when $\sin t = 0$.

$\therefore t = 0, \pi$

Thus, the x-intercepts are: $(1, 0)$ and $(2\pi - 1, 0)$

As shown on the graph, each value of t corresponds to a point whose coordinates are (x, y).

$$S = \int_1^{2\pi - 1} y\,dx$$

$$= \int_0^{\pi} y \frac{dx}{dt} dt$$

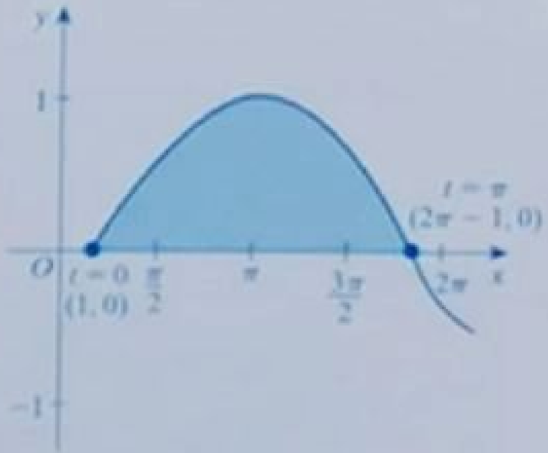
$$= \int_0^{\pi} \sin t \cdot (2 - \sin t) dt$$

$$= \int_0^{\pi} (2\sin t - \sin^2 t) dt$$

$$= \int_0^{\pi} \left(2\sin t - \frac{1 - \cos 2t}{2}\right) dt$$

$$= \left[-2\cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{\pi}$$

$$= 4 - \frac{1}{2}\pi$$



KUMON

O 143 a Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

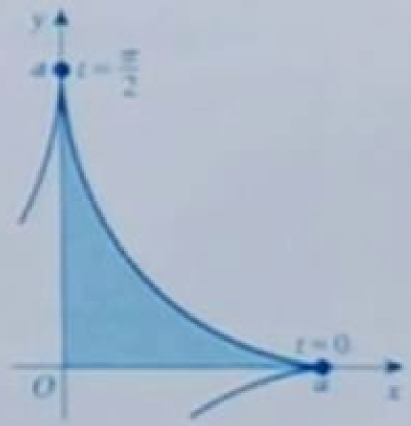
1. Determine the area in the 1st Quadrant bounded by $x = a\cos^3 t$, $y = a\sin^3 t$, the x-axis and the y-axis. $(a > 0)$

$\left(\textbf{Hint:} \text{ When } n \geqq 2 \text{ and even, } \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot n}\cdot\frac{\pi}{2} \right)$

[Sol] As shown on the graph, each value of t corresponds to a point whose coordinates are (x, y).

As x varies from 0 to a,

t varies from $\frac{\pi}{2}$ to 0.



Eliminating t,
$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Letting S be the bounded area,

$$\begin{aligned}
S &= \int_0^a y\,dx \\
&= \int_{\frac{\pi}{2}}^0 y\frac{dx}{dt}\,dt \\
&= \int_{\frac{\pi}{2}}^0 a\sin^3 t(-3a\cos^2 t\sin t)\,dt \\
&= 3a^2\int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt \\
&= 3a^2\int_0^{\frac{\pi}{2}} \sin^4 t(1-\sin^2 t)\,dt \\
&= 3a^2\int_0^{\frac{\pi}{2}} \sin^4 t\,dt - 3a^2\int_0^{\frac{\pi}{2}} \sin^6 t\,dt
\end{aligned}$$

Using the hint,

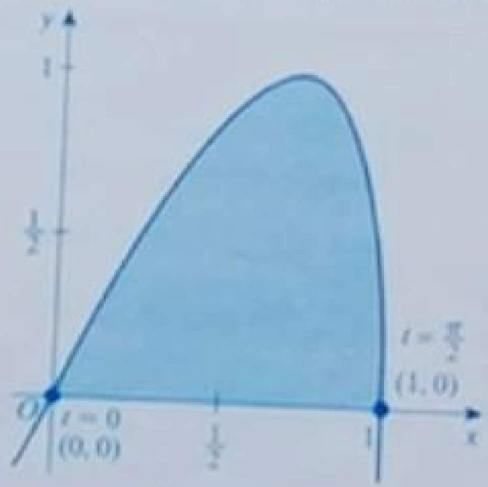
$$\begin{aligned}
S &= 3a^2\cdot\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2}\right) - 3a^2\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{\pi}{2}\right) \\
&= \frac{3}{32}\pi a^2
\end{aligned}$$

O 143 b

2. Determine the area in the 1st Quadrant bounded by $x = \sin t$, $y = 2\sin t\cos t$, the x-axis and the y-axis.

$\left(\textbf{Hint:} \text{ When } n \geq 3 \text{ and odd, } \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \dfrac{2\cdot 4\cdot 6\cdot \ldots \cdot (n-1)}{1\cdot 3\cdot 5\cdot \ldots \cdot n}\right)$

[Sol] As shown on the graph, each value of t corresponds to a point whose coordinates are (x, y).



Eliminating t, $y^2 = 4x^2(1-x^2)$

As x varies from 0 to 1,

t varies from 0 to $\frac{\pi}{2}$.

Letting S be the bounded area,

$$
\begin{aligned}
S &= \int_0^1 y\,dx \\
&= \int_0^{\frac{\pi}{2}} y\frac{dx}{dt}\,dt \\
&= \int_0^{\frac{\pi}{2}} 2\sin t\cos t\cdot\cos t\,dt \\
&= 2\int_0^{\frac{\pi}{2}} \sin t\cos^2 t\,dt \\
&= 2\int_0^{\frac{\pi}{2}} \sin t(1-\sin^2 t)\,dt \\
&= 2\int_0^{\frac{\pi}{2}} \sin t\,dt - 2\int_0^{\frac{\pi}{2}} \sin^3 t\,dt \\
&= -2\Big[\cos t\Big]_0^{\frac{\pi}{2}} - 2\left(\frac{2}{1\cdot 3}\right) \\
&= -2(0-1) - \frac{4}{3} \\
&= \frac{2}{3}
\end{aligned}
$$

KUMON

O 144 a

Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

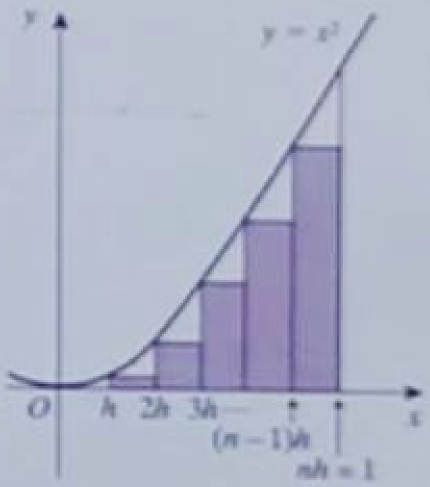
1. Determine the area bounded by the parabola $y = x^2$, the x-axis and the line $x = 1$ by applying the following method.

[Sol] Dividing the interval $0 \leq x \leq 1$ into n equal parts, each subinterval has a width of $\boxed{\frac{1}{n}}$. Letting h represent the width of each subinterval,

Since $h = \boxed{\frac{1}{n}}$, the values of the function at the left side of each subinterval are:

$$0, h^2, (2h)^2, \ldots, [(\boxed{n-1})h]^2$$

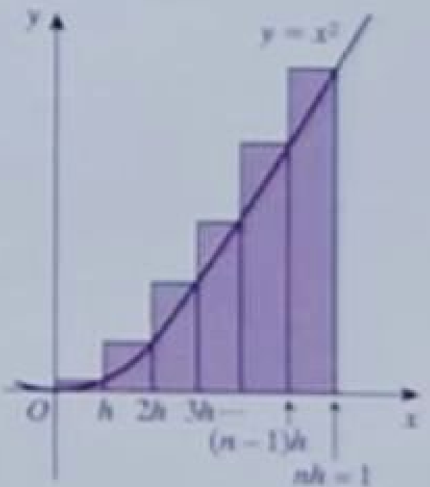
Letting S_n represent the sum of the areas of the shaded rectangles in the graph on the right,



$$S_n = h[0 + h^2 + 2^2h^2 + \cdots + (n-1)^2h^2]$$
$$= h^3[1^2 + 2^2 + \cdots + (n-1)^2]$$

$$\therefore S_n = \frac{1}{n^3} \cdot \boxed{\frac{(n-1)n(2n-1)}{6}}$$

Similarly, letting S'_n represent the sum of the areas of the shaded rectangles in the graph on the right,



$$S'_n = h(h^2 + 2^2h^2 + \cdots + n^2h^2)$$
$$= h^3(1^2 + 2^2 + \cdots + n^2)$$

$$\therefore S'_n = \boxed{\frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}}$$

Therefore,

$$\lim_{n\to\infty} S_n = \frac{1}{3} \text{ and } \lim_{n\to\infty} S'_n = \boxed{\frac{1}{3}}.$$

Thus, the area bounded by the parabola, the x-axis and $x = 1$ is $\boxed{\frac{1}{3}}$.

Determining the value of the area (or volume) under a curve by a series of very fine rectangles and determining the value of the limit is called *Integration by Quadrature*.

O 144 b

2. Using Integration by Quadrature, determine the area bounded by the parabola $y=x^2$, the x-axis and the line $x=a$, where $a>0$.

[Sol] Dividing the interval $0 \le x \le a$ into n equal parts, each subinterval has a width of $\frac{a}{n}$.

$$S_n = h[h^2 + (2h)^2 + \cdots + (nh)^2]$$
$$= h^3(1^2 + 2^2 + \cdots + n^2)$$
$$= \left(\frac{a}{n}\right)^3 \cdot \frac{n(n+1)(2n+1)}{6}$$

$$\therefore S = \lim_{n\to\infty} S_n = \frac{a^3}{3}$$

Alternate Solution:

$$S'_n = h[0 + h^2 + (2h)^2 + \cdots + (n-1)^2h^2]$$
$$= h^3[1^2 + 2^2 + \cdots + (n-1)^2]$$
$$= \left(\frac{a}{n}\right)^3 \cdot \frac{(n-1)n(2n-1)}{6}$$

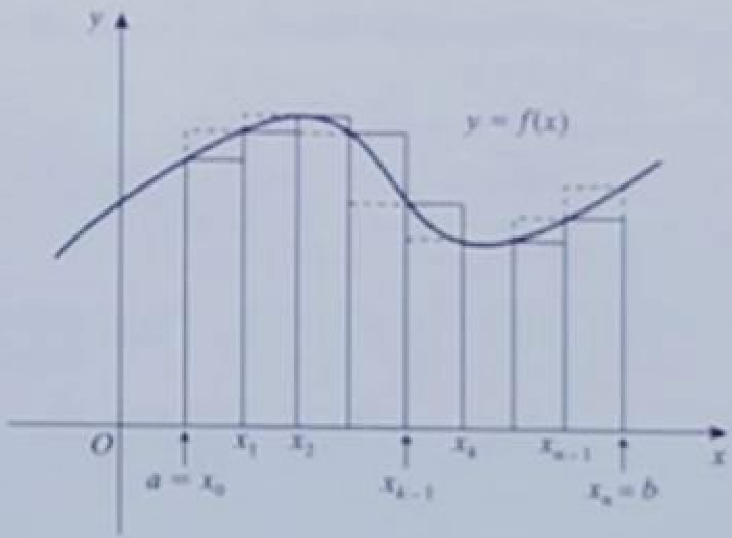
$$\therefore S = \lim_{n\to\infty} S'_n = \frac{a^3}{3}$$

Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

1. Let the function $y = f(x)$ form an unbroken curve over the interval $a \leq x \leq b$, and assume $f(x) > 0$ over this interval. Using Integration by Quadrature, determine the area bounded by $y = f(x)$, the x-axis, $x = a$ and $x = b$.



[Sol] Dividing the interval $a \leq x \leq b$ into n equal parts, let the points of subdivision be:

$$x_0,\ x_1,\ x_2,\ \ldots,\ x_{k-1},\ x_k,\ \ldots,\ x_{n-1},\ x_n$$

$$a = x_0, \qquad\qquad b = x_n$$

Letting h be the width of each subinterval,

$$h = \boxed{\frac{b-a}{n}}$$

Letting S be the bounded area,

$$S = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left[f(\boxed{x_k}) \cdot \boxed{h} \right]$$

Rewriting S,

$$\therefore\ S = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\boxed{f(x_k)} \cdot \boxed{h} \right]$$

O 145 b

2. Using Integration by Quadrature, evaluate each of the following definite integrals.

(1) $\int_0^1 x^2 dx$

[Sol] Dividing the interval $0 \leq x \leq 1$ into n equal parts,

the width of each interval is $h = \frac{1}{n}$.

Letting the subdivision points be $x_k = \frac{k}{n} = hk \quad (k = 0, 1, 2, \dots, n)$,

$$\int_0^1 x^2 dx = \lim_{n\to\infty} \sum_{k=1}^{n} (x_k{}^2 \cdot h)$$

$$\sum_{k=1}^{n} (x_k{}^2 \cdot h) = \sum_{k=1}^{n} [(hk)^2 \cdot h] = h^3 \sum_{k=1}^{n} k^2 = h^3 \cdot \boxed{\frac{n(n+1)(2n+1)}{6}}$$

$$= \frac{1}{6}\left(1 + \boxed{\frac{1}{n}}\right)\left(2 + \boxed{\frac{1}{n}}\right) \quad \left(\text{since } h = \frac{1}{n}\right)$$

$$\therefore \int_0^1 x^2 dx = \lim_{n\to\infty} \left[\frac{1}{6}\left(1 + \boxed{\frac{1}{n}}\right)\left(2 + \boxed{\frac{1}{n}}\right)\right] = \boxed{\frac{1}{3}}$$

(2) $\int_1^3 x^3 dx$

[Sol] Dividing the interval $1 \leq x \leq 3$ into n equal parts,

the width of each interval is $h = \frac{2}{n}$.

Letting the subdivision points be $x_k = 1 + \frac{2k}{n} = 1 + hk \quad (k = 0, 1, 2, \dots, n)$,

$$\int_1^3 x^3 dx = \lim_{n\to\infty} \sum_{k=1}^{n} (x_k{}^3 \cdot h)$$

$$\sum_{k=1}^{n} (x_k{}^3 \cdot h) = \sum_{k=1}^{n} [(1 + hk)^3 \cdot h] = h^4 \sum_{k=1}^{n} k^3 + 3h^3 \sum_{k=1}^{n} k^2 + 3h^2 \sum_{k=1}^{n} k + hn$$

$$= h^4\left[\frac{n(n+1)}{2}\right]^2 + 3h^3 \cdot \frac{n(n+1)(2n+1)}{6} + 3h^2 \cdot \frac{n(n+1)}{2} + hn$$

$$= 4\left(1 + \frac{1}{n}\right)^2 + 4\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 6\left(1 + \frac{1}{n}\right) + 2 \quad \left(\text{since } h = \frac{2}{n}\right)$$

$$\therefore \int_1^3 x^3 dx = 4 + 8 + 6 + 2 = 20$$

KUMON

O 146 a

Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Let $f(x)$ be continuous over $a \leq x \leq b$.
Dividing the interval into n equal parts, where the subdivision points can be expressed as:

$$x_0,\ x_1,\ x_2,\ \dots,\ x_n$$

$$a = x_0, \qquad b = x_n$$

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}[f(x_k)\cdot h] = \lim_{n\to\infty}\sum_{k=1}^{n}[f(x_k)\cdot h] = \int_a^b f(x)dx \qquad \left(\text{where } h = \frac{b-a}{n}\right)$$

Convert each of the following limits into a definite integral and then evaluate the integral.

Ex.

$$\lim_{n\to\infty}\left[\frac{1}{n\sqrt{n}}\left(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)\right] = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\right)$$

Let $a = 0$ and $b = 1$.

$$= \int_0^1 \sqrt{x}\,dx$$

$$= \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$$

(1) $\displaystyle \lim_{n\to\infty}\left(\frac{1}{n^4}\sum_{k=1}^{n}k^3\right)$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3\right]$$

$$= \int_0^1 \boxed{x}^3 dx$$

$$= \frac{1}{4}\left[x^4\right]_0^1 = \frac{1}{4}$$

O 146 b

(2) $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}\right)$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\left(1+\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n-1}{n}}\right)\right]$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{1+\frac{k}{n}}\right)$$

$$=\int_0^1\frac{dx}{1+x}$$

$$=\Big[\ln(1+x)\Big]_0^1$$

$$=\ln 2$$

(3) $\lim_{n\to\infty}\left(\frac{1}{n^2+1^2}+\frac{2}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right)$

$$=\lim_{n\to\infty}\left[\frac{\frac{1}{n^2}}{1+\left(\frac{1}{n}\right)^2}+\frac{\frac{2}{n^2}}{1+\left(\frac{2}{n}\right)^2}+\cdots+\frac{\frac{n}{n^2}}{1+\left(\frac{n}{n}\right)^2}\right]$$

$$=\lim_{n\to\infty}\left\{\frac{1}{n}\left[\frac{\frac{1}{n}}{1+\left(\frac{1}{n}\right)^2}+\frac{\frac{2}{n}}{1+\left(\frac{2}{n}\right)^2}+\cdots+\frac{\frac{n}{n}}{1+\left(\frac{n}{n}\right)^2}\right]\right\}$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\right]$$

$$=\int_0^1\frac{x}{1+x^2}dx$$

$$=\frac{1}{2}\Big[\ln(1+x^2)\Big]_0^1$$

$$=\frac{1}{2}\ln 2$$

Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	-	-	1	2–

Convert each of the following limits into a definite integral and then evaluate the integral.

(1) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+2n}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right)$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\left(\frac{1}{\sqrt{1+\frac{1}{n}}}+\frac{1}{\sqrt{1+\frac{2}{n}}}+\cdots+\frac{1}{\sqrt{1+\frac{n}{n}}}\right)\right]$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\frac{k}{n}}}\right)$$

$$=\int_0^1\frac{dx}{\sqrt{1+x}}$$

$$=2\left[\sqrt{1+x}\right]_0^1$$

$$=2(\sqrt{2}-1)$$

(2) $\lim_{n\to\infty}\left[\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{n\pi}{n}\right)\right]$

$$=\lim_{n\to\infty}\left\{\frac{1}{n}\left[\sin\left(\pi\cdot\frac{1}{n}\right)+\sin\left(\pi\cdot\frac{2}{n}\right)+\cdots+\sin\left(\pi\cdot\frac{n}{n}\right)\right]\right\}$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\sin\left(\frac{k}{n}\pi\right)\right]$$

$$=\int_0^1\sin(\pi x)\,dx$$

$$=-\frac{1}{\pi}\left[\cos(\pi x)\right]_0^1$$

$$=\frac{2}{\pi}$$

O 147 b

(3) $\lim_{n\to\infty}\left[\frac{1}{n}\left(\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\cdots+\cos\frac{n\pi}{n}\right)\right]$

$$=\lim_{n\to\infty}\left\{\frac{1}{n}\left[\cos\left(\pi\cdot\frac{1}{n}\right)+\cos\left(\pi\cdot\frac{2}{n}\right)+\cdots+\cos\left(\pi\cdot\frac{n}{n}\right)\right]\right\}$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\cos\left(\frac{k}{n}\pi\right)\right]$$

$$=\int_0^1\cos(\pi\cdot x)\,dx$$

$$=\frac{1}{\pi}\Big[\sin(\pi x)\Big]_0^1$$

$$=0$$

(4) $\lim_{n\to\infty}\left[\frac{1}{n^2}\left(\cos\frac{\pi}{2n}+2\cos\frac{2\pi}{2n}+\cdots+n\cos\frac{n\pi}{2n}\right)\right]$

$$=\lim_{n\to\infty}\left\{\frac{1}{n}\left[\frac{1}{n}\cos\left(\frac{\pi}{2}\cdot\frac{1}{n}\right)+\frac{2}{n}\cos\left(\frac{\pi}{2}\cdot\frac{2}{n}\right)+\cdots+\frac{n}{n}\cos\left(\frac{\pi}{2}\cdot\frac{n}{n}\right)\right]\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n}\left[\frac{k}{n}\cos\left(\frac{k}{2n}\pi\right)\right]\right\}$$

$$=\int_0^1 x\cos\left(\frac{\pi}{2}x\right)dx$$

$$=\left[x\cdot\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)\right]_0^1-\int_0^1\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)dx$$

$$=\frac{2}{\pi}-\left[-\frac{4}{\pi^2}\cos\left(\frac{\pi}{2}x\right)\right]_0^1$$

$$=\frac{2}{\pi}-\frac{4}{\pi^2}$$

O 148 a Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Convert each of the following limits into a definite integral and then evaluate the integral.

(1) $\lim_{n\to\infty}\left\{\frac{\pi}{n^2}\sum_{k=1}^{n}\left[k\sin\left(\frac{k}{n}\pi\right)\right]\right\}$

$$= \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n}\left[\frac{k}{n}\pi\sin\left(\frac{k}{n}\pi\right)\right]\right\}$$

$$= \int_0^1 x\pi\sin(\pi x)\,dx$$

$$= -\Big[x\cos(\pi x)\Big]_0^1 + \int_0^1 \cos(\pi x)\,dx$$

$$= 1 + \left[\frac{1}{\pi}\sin(\pi x)\right]_0^1$$

$$= 1$$

(2) $\lim_{n\to\infty}\left\{\frac{\pi}{n^2}\sum_{k=1}^{n}\left[(-k)\cos\left(\frac{k}{n}\pi\right)\right]\right\}$

$$= -\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n}\left[\frac{k}{n}\pi\cos\left(\frac{k}{n}\pi\right)\right]\right\}$$

$$= -\int_0^1 x\pi\cos(\pi x)\,dx$$

$$= -\Big[x\sin(\pi x)\Big]_0^1 + \int_0^1 \sin(\pi x)\,dx$$

$$= -(\sin\pi - 0) - \frac{1}{\pi}\Big[\cos(\pi x)\Big]_0^1$$

$$= -\frac{1}{\pi}(-1-1)$$

$$= \frac{2}{\pi}$$

O 148 b

(3) $\lim_{n\to\infty}\left(n\sum_{k=1}^{n}\frac{1}{4n^2-k^2}\right)$

$$=\lim_{n\to\infty}\left\{n\sum_{k=1}^{n}\frac{1}{n^2\left[4-\left(\frac{k}{n}\right)^2\right]}\right\}$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\frac{1}{4-\left(\frac{k}{n}\right)^2}\right]$$

$$=\int_0^1\frac{dx}{4-x^2}$$

$$=\frac{1}{4}\int_0^1\left(\frac{1}{x+2}-\frac{1}{x-2}\right)dx$$

$$=\frac{1}{4}\left[\ln\left|\frac{x+2}{x-2}\right|\right]_0^1$$

$$=\frac{1}{4}\ln 3$$

(4) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(n+k)(3n+k)}$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{k}{n}\right)\left(3+\frac{k}{n}\right)}\right]$$

$$=\int_0^1\frac{1}{(1+x)(3+x)}dx$$

$$=\frac{1}{2}\int_0^1\left(\frac{1}{1+x}-\frac{1}{3+x}\right)dx$$

$$=\frac{1}{2}\left[\ln\left|\frac{1+x}{3+x}\right|\right]_0^1$$

$$=\frac{1}{2}\left(\ln\frac{1}{2}-\ln\frac{1}{3}\right)$$

$$=\frac{1}{2}\ln\frac{3}{2}$$

O 149 a Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

Convert each of the following limits into a definite integral and then evaluate the integral.

(1) $\lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{n^2-k^2}\right)$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n}\sqrt{1-\left(\frac{k}{n}\right)^2}\right]=\int_0^1\sqrt{1-x^2}\,dx$$

Letting $x=\sin\theta$, $dx=\cos\theta\,d\theta$ $\left(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$

When $x=0$, $\theta=0$; when $x=1$, $\theta=\frac{\pi}{2}$.

$$\therefore \lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{n^2-k^2}\right)=\int_0^{\frac{\pi}{2}}\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}=\frac{\pi}{4}$$

(2) $\lim_{n\to\infty}\left(\frac{1}{4n^2}\sum_{k=1}^{n}\sqrt{4n^2-k^2}\right)$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{4n}\sqrt{4-\left(\frac{k}{n}\right)^2}\right]=\frac{1}{4}\int_0^1\sqrt{4-x^2}\,dx$$

Letting $x=2\sin\theta$, $dx=2\cos\theta\,d\theta$ $\left(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$

When $x=0$, $\theta=0$; when $x=1$, $\theta=\frac{\pi}{6}$.

$$\therefore \lim_{n\to\infty}\left(\frac{1}{4n^2}\sum_{k=1}^{n}\sqrt{4n^2-k^2}\right)=\frac{1}{4}\int_0^{\frac{\pi}{6}}2\cos\theta\cdot 2\cos\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{6}}\cos^2\theta\,d\theta=\frac{1}{2}\int_0^{\frac{\pi}{6}}(1+\cos 2\theta)d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}=\frac{\pi}{12}+\frac{\sqrt{3}}{8}$$

O 149 b

(3) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(2n+k)(4n+k)}$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\left(2+\frac{k}{n}\right)\left(4+\frac{k}{n}\right)}\right]$$

$$=\int_0^1\frac{dx}{(2+x)(4+x)}$$

$$=\frac{1}{2}\int_0^1\left(\frac{1}{2+x}-\frac{1}{4+x}\right)dx$$

$$=\frac{1}{2}\left[\ln\left|\frac{2+x}{4+x}\right|\right]_0^1$$

$$=\frac{1}{2}\left(\ln\frac{3}{5}-\ln\frac{1}{2}\right)$$

$$=\frac{1}{2}\ln\frac{6}{5}$$

Applications of Integrals 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

1. Convert each of the following limits into a definite integral and then evaluate the integral.

(1) $\lim_{n\to\infty}\left(\frac{1}{n^5}\sum_{k=1}^{n}k^4\right)$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right]$$

$$=\int_0^1 x^4\,dx$$

$$=\frac{1}{5}\Big[x^5\Big]_0^1$$

$$=\frac{1}{5}$$

(2) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{2n^2+n}}+\frac{1}{\sqrt{2n^2+2n}}+\cdots+\frac{1}{\sqrt{2n^2+n^2}}\right)$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\left(\frac{1}{\sqrt{2+\frac{1}{n}}}+\frac{1}{\sqrt{2+\frac{2}{n}}}+\cdots+\frac{1}{\sqrt{2+\frac{n}{n}}}\right)\right]$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{2+\frac{k}{n}}}\right)$$

$$=\int_0^1\frac{dx}{\sqrt{2+x}}$$

$$=2\Big[\sqrt{2+x}\Big]_0^1$$

$$=2(\sqrt{3}-\sqrt{2})$$

O 150 b

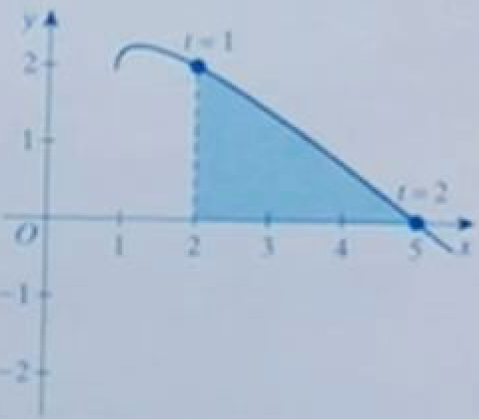
2. Given that for each real value of t, there is a point with coordinates (x, y), where $x = 1 + t^2$ and $y = 2 + t - t^2$, let l be the line formed by the set of these (x, y) points. Determine the area bounded by line l, the x-axis and the line $x = 2$.

[Sol] When $y = 0$, $t = 2$.
When $x = 2$, $t = 1$.

As shown on the graph, each value of t corresponds to a point whose coordinates are (x, y).

Thus, the curve from $t = 1$ to $t = 2$ and the x-axis bound the shaded area.

$$
\begin{aligned}
S &= \int_1^2 y\frac{dx}{dt}\,dt \\
&= \int_1^2 (2 + t - t^2)(2t)\,dt \\
&= 2\int_1^2 (2t + t^2 - t^3)\,dt \\
&= 2\left[t^2 + \frac{t^3}{3} - \frac{t^4}{4}\right]_1^2 \\
&= 2\left(\frac{19}{12}\right) \\
&= \frac{19}{6}
\end{aligned}
$$



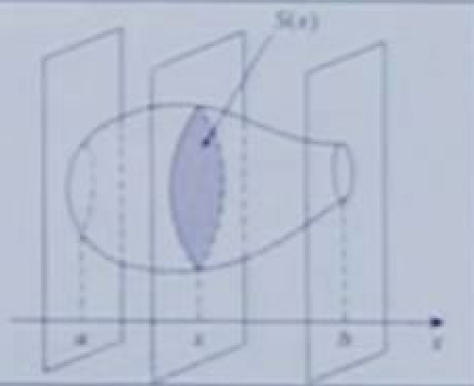
O 151 a Applications of Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

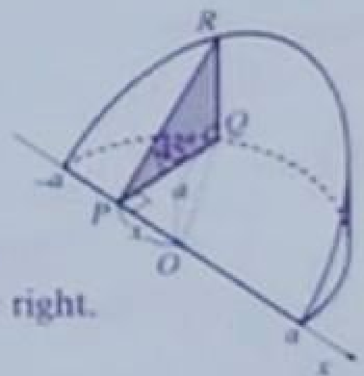
Let V be the volume of the solid between the planes $x=a$ and $x=b$, where $a<b$. Letting $S(x)$ be the area of the intersection of the plane at x with the solid,

$$V=\int_a^b S(x)\,dx$$



1. Given a right circular cylinder of radius a, a wedge is cut off by a plane through the center of the base at a 45° angle to it. Determine the volume of this wedge.

[Sol] Letting O be the center of the base, the x-axis the intersection of the plane and the base, and $P(x, 0)$ the point where $S(x)$ passes through the x-axis, $S(x)$ is the area of the shaded rectangular isosceles $\triangle PQR$ ($\angle Q=90°$) shown at the right.



$$\therefore\ S(x)=\frac{1}{2}PQ\cdot QR=\frac{1}{2}PQ^2$$

$$=\frac{1}{2}(\sqrt{a^2-x^2})^2=\frac{1}{2}(a^2-x^2)$$

Letting V be the volume,

$$V=\int_{-a}^{a} S(x)dx$$

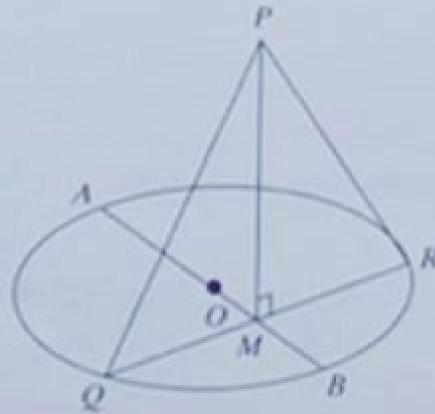
Note from the graph that the volume from $-a$ to 0 is symmetric to the volume from 0 to a.

$$\therefore\ V=2\int_0^a S(x)\,dx$$

$$=\int_0^a (a^2-x^2)\,dx=\left[a^2x-\frac{x^3}{3}\right]_0^a=a^3-\frac{a^3}{3}=\frac{2}{3}a^3$$

O 151 b

2. AB is a diameter of a circle with center O and a radius a. QR is a chord perpendicular to AB. $\triangle PQR$ (where $PQ = PR$) is an isosceles triangle in a plane perpendicular to AB. PM is the altitude of the triangle. Determine the volume generated by $\triangle PQR$ as M moves from A to B, where $PM + OM = a$.



[Sol] Letting O be the origin,

OB the x-axis, and $OM = x$,

$PM = a - x$ and

$QM = RM = \sqrt{a^2 - x^2}$

$\therefore$ The area of $\triangle PQR = \frac{1}{2} PM \cdot QR$

$$= \frac{1}{2}(a-x) \cdot 2\sqrt{a^2 - x^2} = (a-x)\sqrt{a^2 - x^2}$$

Letting V be the volume,

$$V = \int_{-a}^{a} (a-x)\sqrt{a^2 - x^2}\,dx$$

$$= 2\int_{0}^{a} (a-x)\sqrt{a^2 - x^2}\,dx \qquad \text{(from symmetry)}$$

Letting $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$ $\quad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$

When $x = 0$, $\theta = 0$; when $x = a$, $\theta = \frac{\pi}{2}$

Since $x = a\sin\theta$, $\sqrt{a^2 - x^2} = a\cos\theta$

$$\therefore V = 2\int_{0}^{\frac{\pi}{2}} a(1 - \sin\theta) \cdot a\cos\theta \cdot a\cos\theta\,d\theta$$

$$= 2a^3 \int_{0}^{\frac{\pi}{2}} (\cos^2\theta - \cos^2\theta\sin\theta)\,d\theta$$

$$= 2a^3 \left(\int_{0}^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2}\,d\theta - \int_{0}^{\frac{\pi}{2}} \cos^2\theta\sin\theta\,d\theta \right)$$

$$= 2a^3 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{0}^{\frac{\pi}{2}} + 2a^3 \left[\frac{\cos^3\theta}{3} \right]_{0}^{\frac{\pi}{2}}$$

$$= 2a^3 \left(\frac{\pi}{4} - \frac{1}{3} \right) = \frac{a^3(3\pi - 4)}{6}$$

O 152 a Applications of Integrals 3

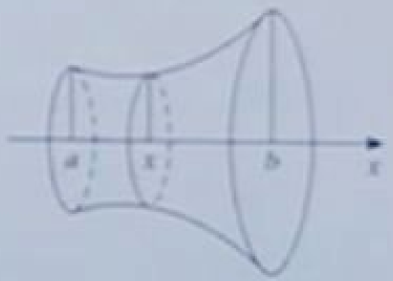
Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

Rotating $y = f(x)$ $(a \leq x \leq b)$ around the x-axis once forms a solid shape.
Letting $|f(x)|$ be the radius at x, a circle forms.
Letting $S(x)$ be the area of the circle,

$$S(x) = \pi|f(x)|^2 = \pi[f(x)]^2$$

Therefore, letting V be the volume of the solid shape,



$$V(x) = \pi\int_a^b [f(x)]^2 dx$$

Similarly,

Rotating $x = f(y)(a \leq y \leq b)$ around the y-axis once forms a solid shape.
Letting $|f(y)|$ be the radius at y, a circle forms.
Letting $S(y)$ be the area of the circle,

$$S(y) = \pi|f(y)|^2 = \pi[f(y)]^2$$

Therefore, letting V be the volume of the solid shape,

$$V(y) = \pi\int_a^b [f(y)]^2 dy$$

1. Determine the volume of the solid generated when the part of the curve in the 1st Quadrant of $y = 1 - \sqrt{x}$ is rotated around the x-axis.

[Sol] Since the function intersects the y-axis at $x = 0$ and intersects the x-axis at $x = 1$,

$$V = \pi\int_0^1 (1 - \sqrt{x})^2 dx$$

$$= \pi\int_0^1 (1 - 2x^{\frac{1}{2}} + x)\,dx$$

$$= \pi\left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2\right]_0^1$$

$$= \frac{\pi}{6}$$

2. Given the region bounded by $y = \ln x$, its tangent line which passes through the origin, and the x-axis, complete the following exercises.

(a) Determine the equation of the tangent line.

[Sol] Since $y' = \frac{1}{x}$, the tangent line at $(x_1, \ln x_1)$ is $y - \ln x_1 = \frac{1}{x_1}(x - x_1)$.

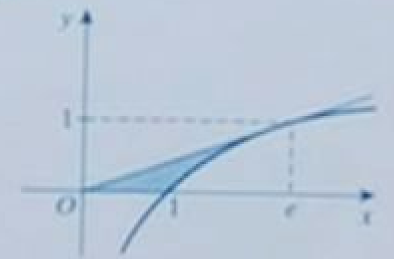
Since the tangent line passes through the origin,

$$-\ln x_1 = -1$$

$$\therefore x_1 = e$$

Thus, the equation of the tangent line is:

$$y = \frac{1}{e}x$$



(b) Determine the volume of the solid when it is rotated around the x-axis.

[Sol] Letting V be the volume,

$$V = \pi\int_0^e \left(\frac{1}{e}x\right)^2 dx - \pi\int_1^e (\ln x)^2 dx$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2x\ln x \cdot \frac{1}{x}dx$$
$$= x(\ln x)^2 - 2(x\ln x - x) + C$$
$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

$$\therefore V = \pi\left[\frac{1}{3e^2}x^3\right]_0^e - \pi\left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^e$$
$$= \pi\frac{e}{3} - \pi(e - 2e + 2e - 2) = \pi\left(2 - \frac{2}{3}e\right)$$

(c) Determine the volume of the solid when it is rotated around the y-axis.

[Sol] Rewriting $y = \ln x$, $\quad x = \boxed{e^y}$.

Rewriting $y = \frac{1}{e}x$, $\quad x = \boxed{ey}$.

Thus, $V = \pi\int_0^1 (e^y)^2 dy - \pi\int_0^1 (ey)^2 dy$

$$= \pi\left[\frac{1}{2}e^{2y}\right]_0^1 - \pi e^2\left[\frac{1}{3}y^3\right]_0^1$$
$$= \frac{\pi}{2}e^2 - \frac{\pi}{2} - \frac{1}{3}\pi e^2 = \frac{\pi}{6}(e^2 - 3)$$

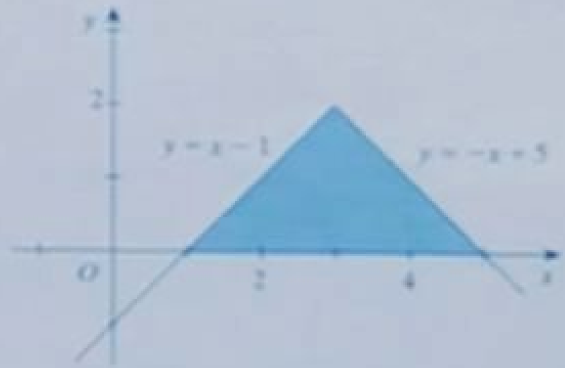
O 153 a Applications of Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

1. Determine the volume generated when the region bounded by the curve $y = 2 - |x - 3|$ and the x-axis is rotated around the x-axis.

[Sol] When $x < 3$, $\quad y = 2 + (x - 3) = x - 1$
When $x \geqq 3$, $\quad y = 2 - (x - 3) = -x + 5$



Letting V be the volume,

$$V = \pi \int_1^3 (x - 1)^2 dx + \pi \int_3^5 (-x + 5)^2 dx$$

$$= \pi \int_1^3 (x^2 - 2x + 1) dx + \pi \int_3^5 (x^2 - 10x + 25) dx$$

$$= \pi \left[\frac{1}{3}x^3 - x^2 + x \right]_1^3 + \pi \left[\frac{1}{3}x^3 - 5x^2 + 25x \right]_3^5$$

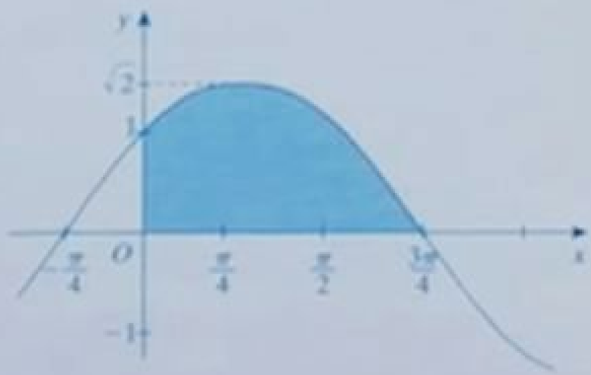
$$= \pi \left[(9 - 9 + 3) - \left(\frac{1}{3} - 1 + 1 \right) \right] + \pi \left[\left(\frac{125}{3} - 125 + 125 \right) - (9 - 45 + 75) \right]$$

$$= \frac{16\pi}{3}$$

O 153 b

2. Determine the volume generated when the region bounded by the curve $y = \sin x + \cos x$ $\left(0 \leq x \leq \frac{3\pi}{4}\right)$ and the x-axis is rotated around the x-axis.

[Sol] Letting V be the volume,



$$V = \pi \int_0^{\frac{3\pi}{4}} (\sin x + \cos x)^2 \, dx$$

$$= \pi \int_0^{\frac{3\pi}{4}} (1 + 2\sin x \cos x) \, dx$$

$$= \pi \left[x + \sin^2 x \right]_0^{\frac{3\pi}{4}}$$

$$= \pi \left[\left(\frac{3\pi}{4} + \frac{1}{2} \right) - 0 \right]$$

$$= \frac{3}{4}\pi^2 + \frac{1}{2}\pi$$

$$\left[= \frac{\pi}{4}(3\pi + 2) \right]$$

KUMON

O 154 a Applications of Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

1. When the circle $x^2+(y-a)^2=r^2$ is rotated around the x-axis, the surface generated is called a torus. Determine its volume when $0<r\leq a$.

[Sol] The circle $x^2+(y-a)^2=r^2$ is formed by an upper and lower part, whose equations are:

$y=a+\sqrt{\boxed{r^2-x^2}}$ ...①, and

$y=a-\sqrt{\boxed{r^2-x^2}}$ ...②, respectively.

Letting V be the volume of the torus,

$$V=\pi\int_{-r}^{r}(a+\sqrt{r^2-x^2})^2dx-\pi\int_{-r}^{r}(a-\sqrt{r^2-x^2})^2dx$$

$$=2\pi\int_0^r[(a+\sqrt{r^2-x^2})^2-(a-\sqrt{r^2-x^2})^2]dx$$

$$=2\pi\int_0^r 4a\sqrt{r^2-x^2}\,dx$$

$$=8a\pi\int_0^r\sqrt{r^2-x^2}\,dx$$

Letting $x=r\sin t$, $dx=r\cos t\,dt$ $\left(-\frac{\pi}{2}\leq t\leq\frac{\pi}{2}\right)$

When $x=0, t=0$; when $x=r, t=\frac{\pi}{2}$

$$\therefore V=8a\pi\int_0^{\frac{\pi}{2}}r\cos t\cdot r\cos t\,dt$$

$$=8ar^2\pi\int_0^{\frac{\pi}{2}}\cos^2t\,dt$$

$$=8ar^2\pi\int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}dt$$

$$=4ar^2\pi\left[t+\frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}}=2ar^2\pi^2$$

O 154 b

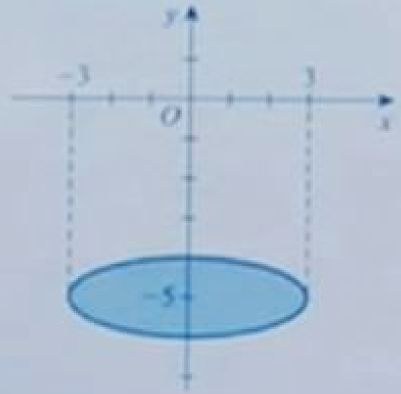
2. Determine the volume generated when the ellipse $\frac{x^2}{9}+y^2=1$ is rotated around the line $y=5$.

[Sol] Rewriting $\frac{x^2}{9}+y^2=1$,

$$y=\pm\frac{1}{3}\sqrt{9-x^2}$$

Translating the ellipse -5 units along the y-axis,

$$y=-5\pm\frac{1}{3}\sqrt{9-x^2}$$



Letting V be the volume,

$$V=\pi\int_{-3}^{3}\left[\left(-5-\frac{1}{3}\sqrt{9-x^2}\right)^2-\left(-5+\frac{1}{3}\sqrt{9-x^2}\right)^2\right]dx$$

$$=2\pi\int_0^3\frac{20}{3}\sqrt{9-x^2}\,dx=\frac{40\pi}{3}\int_0^3\sqrt{9-x^2}\,dx$$

Letting $x=3\sin t$, $dx=3\cos t\,dt$ $\left(-\frac{\pi}{2}\le t\le\frac{\pi}{2}\right)$

When $x=0, t=0$; when $x=3, t=\frac{\pi}{2}$

$$\therefore V=\frac{40\pi}{3}\int_0^{\frac{\pi}{2}}3\cos t\cdot 3\cos t\,dt$$

$$=120\pi\int_0^{\frac{\pi}{2}}\cos^2 t\,dt$$

$$=120\pi\int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\,dt$$

$$=60\pi\left[t+\frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}}$$

$$=30\pi^2$$

O 155 a Applications of Integrals 3

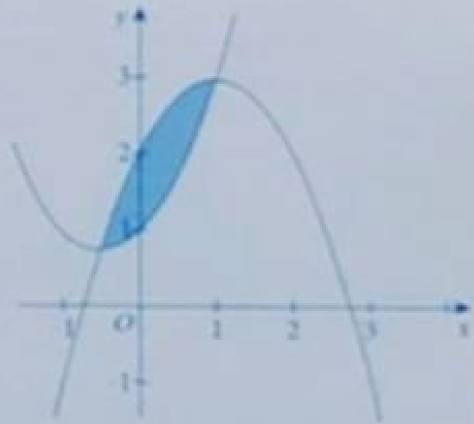
Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	–	1–

1. Determine the volume generated when the region bounded by the curves $y=x^2+x+1$ and $y=-x^2+2x+2$ is rotated around the x-axis.

[Sol] Finding the points of intersection.

$$x^2+x+1=-x^2+2x+2$$
$$2x^2-x-1=0$$
$$(2x+1)(x-1)=0$$
$$x=-\frac{1}{2},\ 1$$



Letting V be the volume,

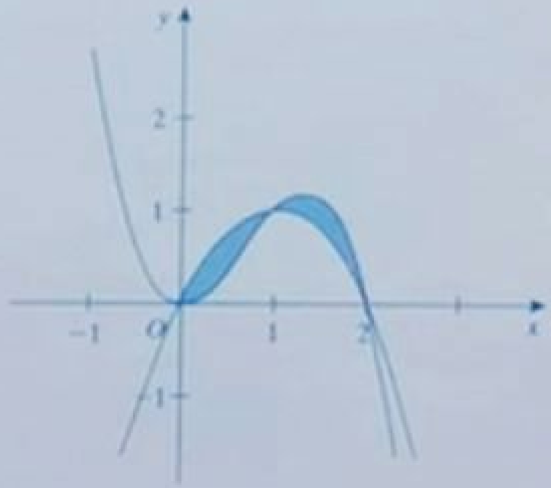
$$V=\pi\int_{-\frac{1}{2}}^{1}[(-x^2+2x+2)^2-(x^2+x+1)^2]dx$$
$$=\pi\int_{-\frac{1}{2}}^{1}(-6x^3-3x^2+6x+3)dx$$
$$=\pi\left[-\frac{3}{2}x^4-x^3+3x^2+3x\right]_{-\frac{1}{2}}^{1}$$
$$=\pi\left[\left(-\frac{3}{2}-1+3+3\right)-\left(-\frac{3}{32}+\frac{1}{8}+\frac{3}{4}-\frac{3}{2}\right)\right]$$
$$=\frac{135\pi}{32}$$

O 155 b

2. Determine the volume generated when the region bounded by the curves $y = x(2-x)$ and $y = x^2(2-x)$ is rotated around the x-axis.

[Sol] Finding the points of intersection.

$$
\begin{aligned}
2x - x^2 &= 2x^2 - x^3 \\
x^3 - 3x^2 + 2x &= 0 \\
x(x-2)(x-1) &= 0 \\
x &= 0, 1, 2
\end{aligned}
$$



Letting V be the volume,

$$
\begin{aligned}
V &= \pi \int_0^1 [(2x-x^2)^2 - (2x^2-x^3)^2]dx + \pi \int_1^2 [(2x^2-x^3)^2 - (2x-x^2)^2]dx \\
&= \pi \int_0^1 (4x^2 - 4x^3 - 3x^4 + 4x^5 - x^6)dx + \pi \int_1^2 (x^6 - 4x^5 + 3x^4 + 4x^3 - 4x^2)dx \\
&= \pi \left[\frac{4}{3}x^3 - x^4 - \frac{3}{5}x^5 + \frac{2}{3}x^6 - \frac{1}{7}x^7\right]_0^1 + \pi \left[\frac{1}{7}x^7 - \frac{2}{3}x^6 + \frac{3}{5}x^5 + x^4 - \frac{4}{3}x^3\right]_1^2 \\
&= \pi\left(\frac{4}{3} - 1 - \frac{3}{5} + \frac{2}{3} - \frac{1}{7}\right) + \pi\left[\left(\frac{128}{7} - \frac{128}{3} + \frac{96}{5} + 16 - \frac{32}{3}\right) - \left(\frac{1}{7} - \frac{2}{3} + \frac{3}{5} + 1 - \frac{4}{3}\right)\right] \\
&= \frac{2\pi}{3}
\end{aligned}
$$

KUMON

O 156 a Applications of Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

1. Determine the volume generated when the region bounded by the curve $y=\sqrt{x}\sin x^2$ $(0\leq x\leq\sqrt{\pi})$ and the x-axis is rotated around the x-axis.

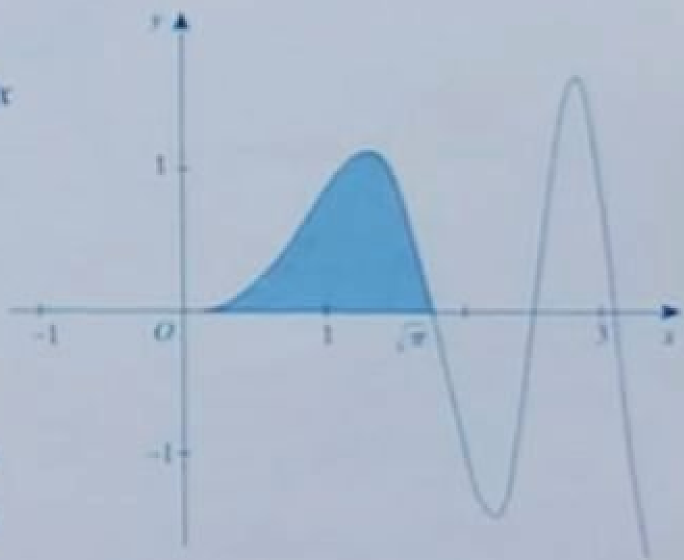
[Sol] $\sin x^2=0$ when $x^2=0, \pi, 2\pi, \ldots$

or when $x=0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots$

Letting V be the volume,

$$V=\pi\int_0^{\sqrt{\pi}}\left[\sqrt{x}\sin x^2\right]^2dx$$

$$=\pi\int_0^{\sqrt{\pi}}x\sin^2 x^2dx$$



Letting $u=x^2$

$du=2x\,dx$

When $x=0,$ $u=0;$

When $x=\sqrt{\pi},$ $u=\pi$

$$\therefore V=\frac{\pi}{2}\int_0^{\pi}\sin^2 u\,du$$

$$=\frac{\pi}{2}\int_0^{\pi}\left[\frac{1-\cos 2u}{2}\right]du$$

$$=\frac{\pi}{4}\left[u-\frac{1}{2}\sin 2u\right]_0^{\pi}$$

$$=\frac{\pi}{4}[(\pi-0)-(0-0)]$$

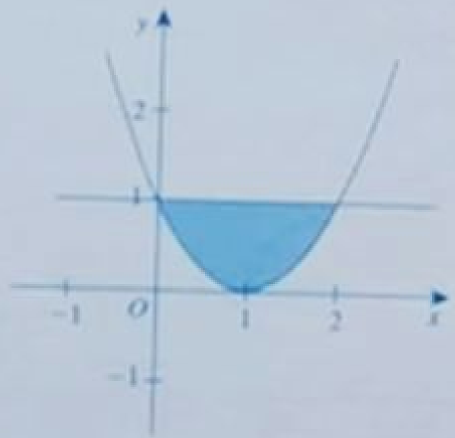
$$=\frac{\pi^2}{4}$$

O 156 b

2. Determine the volume generated when the region bounded by the curve $y = x^2 - 2x + 1$ and the line $y = 1$ is rotated around the x-axis.

[Sol] Finding the points of intersection,

$$x^2 - 2x + 1 = 1$$
$$x(x-2) = 0$$
$$x = 0, 2$$



Letting V be the volume,

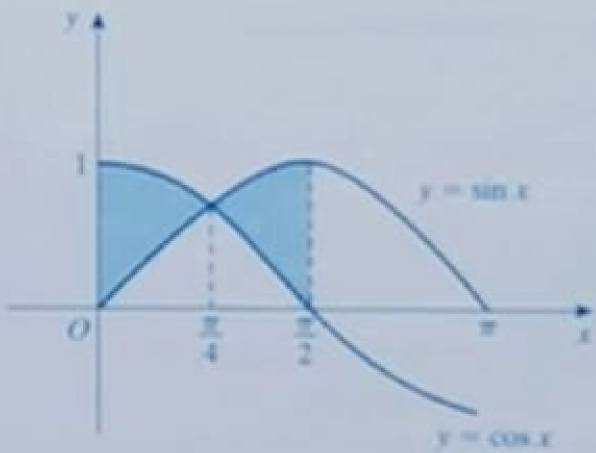
$$V = \pi \int_0^2 [(1)^2 - (x^2 - 2x + 1)^2] dx$$
$$= \pi \int_0^2 (-x^4 + 4x^3 - 6x^2 + 4x)\, dx$$
$$= \pi \left[-\frac{1}{5}x^5 + x^4 - 2x^3 + 2x^2 \right]_0^2$$
$$= \pi \left[\left(-\frac{32}{5} + 16 - 16 + 8 \right) - 0 \right]$$
$$= \frac{8\pi}{5}$$

O 157 a Applications of Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	[illegible]–

1. The two curves $y = \sin x$ and $y = \cos x$ bounded by $x = 0$ and $x = \dfrac{\pi}{2}$ form a region. Determine the volume generated when the region is rotated around the x-axis.



[Sol] Letting V be the volume,

$$V = \pi\int_0^{\frac{\pi}{4}}(\cos^2 x - \sin^2 x)\,dx + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin^2 x - \cos^2 x)\,dx$$

$$= \pi\int_0^{\frac{\pi}{4}}\cos 2x\,dx + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(-\cos 2x)\,dx$$

$$= \frac{\pi}{2}\Big[\sin 2x\Big]_0^{\frac{\pi}{4}} - \frac{\pi}{2}\Big[\sin 2x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

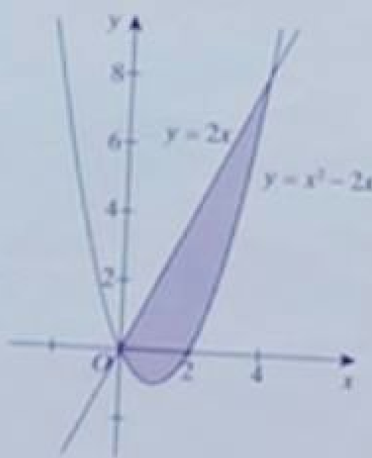
O 157 b

> If a region being rotated is partly above and partly below the axis of rotation, then when the region is rotated, there will be an overlapping part. Any part that causes overlapping can be disregarded.

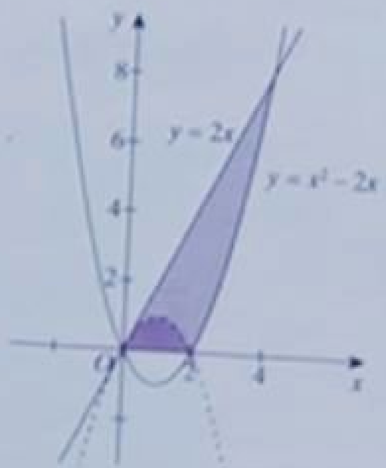
2. Determine the volume generated when the region bounded by the curve $y=x^2-2x$ and the line $y=2x$ is rotated around the x-axis.

[Sol] Finding the points of intersection,

$$\begin{aligned} x^2-2x&=2x \\ x^2-4x&=0 \\ x(x-4)&=0 \\ x&=0,\,4 \end{aligned}$$



When the region is rotated around the x-axis, there is an overlapping part. This part, below the x-axis, that causes overlapping can be disregarded.



Letting V be the required volume,

$$V=\pi\int_0^4 (2x)^2dx-\pi\int_{\boxed{2}}^4 (x^2-2x)^2dx$$

$$=\pi\int_0^4 4x^2dx-\pi\int_2^4 (x^4-4x^3+4x^2)\,dx$$

$$=\pi\left[\frac{4}{3}x^3\right]_0^4-\pi\left[\frac{x^5}{5}-x^4+\frac{4}{3}x^3\right]_2^4$$

$$=\pi\left(\frac{256}{3}-0\right)-\pi\left[\left(\frac{1024}{5}-256+\frac{256}{3}\right)-\left(\frac{32}{5}-16+\frac{32}{3}\right)\right]$$

$$=\frac{784\pi}{15}$$

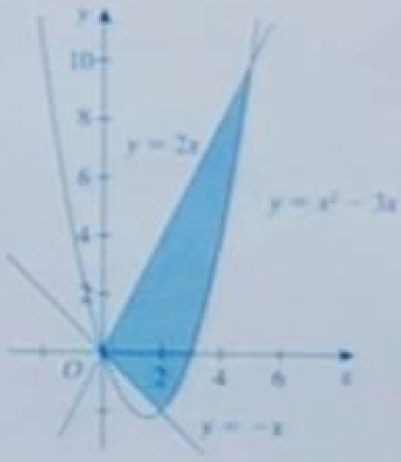
O 158 a Applications of Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

1. Determine the volume generated when the region bounded by the curve $y = x^2 - 3x$ and the lines $y = 2x$ and $y = -x$ is rotated around the x-axis.

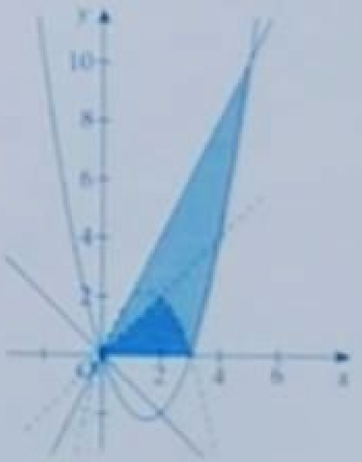
[Sol] Finding the points of intersection.



$$x^2 - 3x = 2x$$
$$x^2 - 5x = 0$$
$$x(x-5) = 0$$
$$x = 0, 5$$

$$x^2 - 3x = -x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0, 2$$

When the region is rotated around the x-axis there is an overlapping part. This overlapping part can be disregarded.



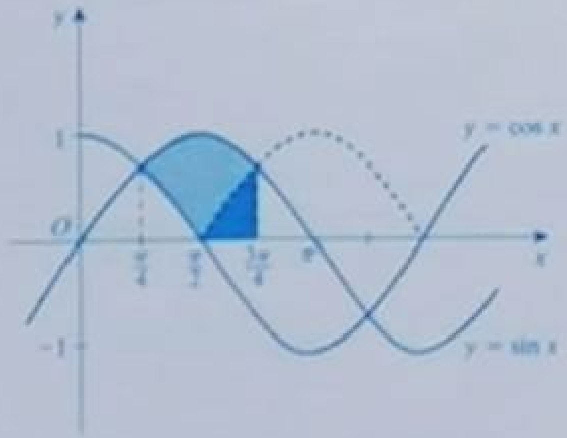
Letting V be the required volume,

$$V = \pi\int_0^5 (2x)^2 dx - \pi\int_3^5 (x^2-3x)^2 dx$$
$$= \pi\int_0^5 4x^2 dx - \pi\int_3^5 (x^4 - 6x^3 + 9x^2)dx$$
$$= \pi\left[\frac{4}{3}x^3\right]_0^5 - \pi\left[\frac{x^5}{5} - \frac{3}{2}x^4 + 3x^3\right]_3^5$$
$$= \pi\left(\frac{500}{3} - 0\right) - \pi\left[\left(625 - \frac{1875}{2} + 375\right) - \left(\frac{243}{5} - \frac{243}{2} + 81\right)\right]$$
$$= \frac{1684\pi}{15}$$

O 158 b

2. The two curves $y = \sin x$ and $y = \cos x$ bounded by $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ form a region. Determine the volume generated when the region is rotated around the x-axis.

[Sol] When the region is rotated around the x-axis there is an overlapping part. This overlapping part can be disregarded.



Letting V be the required volume,

$$V = \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^2 x\,dx - \pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos^2 x\,dx$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1-\cos 2x}{2}\,dx - \pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{1+\cos 2x}{2}\,dx$$

$$= \pi\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} - \pi\left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}$$

$$= \frac{1}{8}\pi(\pi + 6)$$

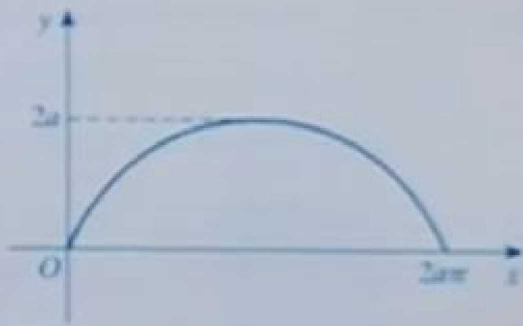
Applications of Integrals 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

1. Given the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, where $0 \leq \theta \leq 2\pi$ and $a > 0$, determine the volume generated when the cycloid is rotated around the x-axis.

[Sol] Letting V be the volume,



$$V = \pi\int_0^{2a\pi} y^2 dx$$

$$= \pi\int_0^{2\pi} y^2\frac{dx}{d\theta}d\theta$$

$$= \pi\int_0^{2\pi} a^2(1-\cos\theta)^2 \cdot a(1-\cos\theta)\,d\theta$$

$$= a^3\pi\int_0^{2\pi}(1-\cos\theta)^3 d\theta$$

$$= 2a^3\pi\int_0^{\pi}(1-\cos\theta)^3 d\theta$$

$$= 2a^3\pi\int_0^{\pi}(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)\,d\theta$$

$$= 2a^3\pi\Big[\theta\Big]_0^{\pi} - 6a^3\pi\Big[\sin\theta\Big]_0^{\pi} + 6a^3\pi\int_0^{\pi}\frac{1+\cos 2\theta}{2}d\theta$$

$$- 2a^3\pi\int_0^{\pi}\cos\theta\cdot\cos^2\theta\, d\theta$$

$$= 2a^3\pi^2 - 0 + 3a^3\pi\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\pi} - 2a^3\pi\int_0^{\pi}\cos\theta\cdot(1-\sin^2\theta)\,d\theta$$

$$= 2a^3\pi^2 + 3a^3\pi^2 - 2a^3\pi\int_0^{\pi}\cos\theta\, d\theta + 2a^3\pi\int_0^{\pi}\cos\theta\sin^2\theta\, d\theta$$

$$= 5a^3\pi^2 - 0 + 0$$

$$= 5a^3\pi^2$$

O 159 b

2. Given: $x = \tan\theta$ and $y = \cos 2\theta$, where $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

(a) Find the coordinates of the points where the graph intersects the x-axis.

[Sol] When $y = 0$, $\cos 2\theta = 0$. $\quad \therefore 2\theta = \pm\frac{\pi}{2} \quad \therefore \theta = \pm\frac{\pi}{4}$

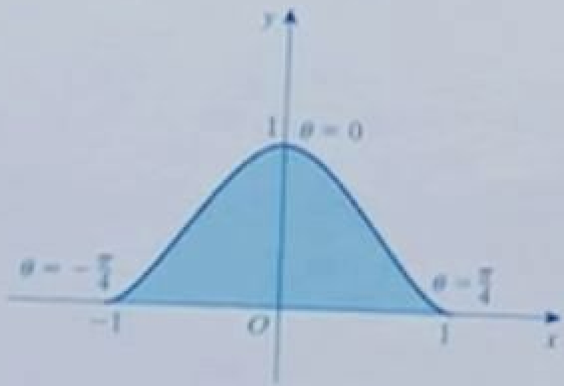
Therefore, $x = \tan\theta = \pm 1$

Thus, the x-axis intersections are: $(-1, 0)$ and $(1, 0)$

(b) Find the volume generated when the curve is rotated around the x-axis.

[Sol] The graph of the curve is shown below.

Letting V be the volume,



$$V = \pi\int_{-1}^{1} y^2 dx$$

$$= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y^2 \frac{dx}{d\theta} d\theta$$

$$= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \cdot \frac{1}{\cos^2\theta} d\theta$$

$$= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 2\theta}{\cos^2\theta} d\theta$$

$$= 2\pi\int_{0}^{\frac{\pi}{4}} \frac{\cos^2 2\theta}{\cos^2\theta} d\theta$$

Since $\dfrac{\cos^2 2\theta}{\cos^2\theta} = \dfrac{(2\cos^2\theta - 1)^2}{\cos^2\theta} = 4\cos^2\theta - 4 + \dfrac{1}{\cos^2\theta}$,

$$V = 2\pi\int_{0}^{\frac{\pi}{4}}\left(4\cos^2\theta - 4 + \frac{1}{\cos^2\theta}\right)d\theta$$

$$= 2\pi\int_{0}^{\frac{\pi}{4}}\left(2\cos 2\theta + \frac{1}{\cos^2\theta} - 2\right)d\theta$$

$$= 2\pi\left[\sin 2\theta + \tan\theta - 2\theta\right]_{0}^{\frac{\pi}{4}}$$

$$= 2\pi\left(1 + 1 - \frac{\pi}{2}\right)$$

$$= \pi(4 - \pi)$$

O 160 a Applications of Integrals 3

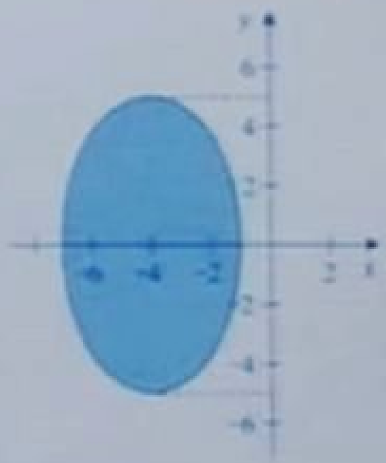
Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Determine the volume generated when the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$ is rotated around the line $x=4$.

[Sol] Rewriting $\frac{x^2}{9}+\frac{y^2}{25}=1$,

$$x=\pm\frac{3}{5}\sqrt{25-y^2}$$



Translating the ellipse -4 units along the x-axis,

$$x=-4\pm\frac{3}{5}\sqrt{25-y^2}$$

Letting V be the volume,

$$V=\pi\int_{-5}^{5}\left[\left(-4-\frac{3}{5}\sqrt{25-y^2}\right)^2-\left(-4+\frac{3}{5}\sqrt{25-y^2}\right)^2\right]dy$$

$$=2\pi\int_0^5\frac{48}{5}\sqrt{25-y^2}\,dy$$

Letting $y=5\sin\theta$, $dy=5\cos\theta\,d\theta$ $\left(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$

When $y=0$, $\theta=0$; when $y=5$, $\theta=\frac{\pi}{2}$

$$\therefore V=\frac{96}{5}\pi\int_0^{\frac{\pi}{2}}(5\cos\theta)(5\cos\theta)\,d\theta$$

$$=480\pi\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$=240\pi\int_0^{\frac{\pi}{2}}(1+\cos2\theta)\,d\theta$$

$$=240\pi\left[\theta+\frac{1}{2}\sin2\theta\right]_0^{\frac{\pi}{2}}$$

$$=120\pi^2$$

O 160 b

2. Determine the volume generated when the region bounded by the curves $y = \sin x + 1$ and $y = \cos x + 1$ $(0 \leq x \leq 2\pi)$ is rotated around the x-axis.

[Sol] Finding the points of intersection.

$$\sin x + 1 = \cos x + 1$$

$$\sin x = \cos x$$

This occurs at

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Letting V be the volume,

$$V = \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [(\sin x + 1)^2 - (\cos x + 1)^2] dx$$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin^2 x - \cos^2 x + 2\sin x - 2\cos x)\, dx$$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (-\cos 2x + 2\sin x - 2\cos x)\, dx$$

$$= \pi \left[-\frac{1}{2}\sin 2x - 2\cos x - 2\sin x \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \pi \left[\left(-\frac{1}{2} + \sqrt{2} + \sqrt{2} \right) - \left(-\frac{1}{2} - \sqrt{2} - \sqrt{2} \right) \right]$$

$$= 4\sqrt{2}\pi$$

O 161 a Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

Letting L be the length of the curve $y = f(x)$ (where $a \leq x \leq b$),

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

In each exercise, determine the length of the given curve.

Ex. $y = \frac{2}{3}\sqrt{x^3}$ $(0 \leq x \leq 3)$

[Sol] $\frac{dy}{dx} = x^{\frac{1}{2}}$

Letting L be the length of the curve,

$$L = \int_0^3 \sqrt{1 + (x^{\frac{1}{2}})^2}\,dx = \int_0^3 \sqrt{1 + x}\,dx$$

$$= \frac{2}{3}\left[(1 + x)^{\frac{3}{2}}\right]_0^3$$

$$= \frac{14}{3}$$

(1) $y = \frac{2}{3}\sqrt{(x-1)^3}$ $(1 \leq x \leq 9)$

[Sol] $\frac{dy}{dx} = (x-1)^{\frac{1}{2}}$

Letting L be the length of the curve,

$$L = \int_1^9 \sqrt{1 + [(x-1)^{\frac{1}{2}}]^2}\,dx = \int_1^9 \sqrt{x}\,dx$$

$$= \frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^9$$

$$= \frac{52}{3}$$

O 161 b

(2) $y=\frac{1}{8}x^4+\frac{1}{4x^2}$ $(1\le x\le 2)$

[Sol] $y=\frac{1}{8}x^4+\frac{1}{4}x^{-2}$

$$\frac{dy}{dx}=\frac{1}{2}x^3-\frac{1}{2}x^{-3}=\frac{1}{2}(x^3-x^{-3})$$

Letting L be the length of the curve,

$$L=\int_1^2\sqrt{1+\left(\frac{1}{4}x^6-\frac{1}{2}+\frac{1}{4}x^{-6}\right)}\,dx=\int_1^2\sqrt{\frac{1}{4}x^6+\frac{1}{2}+\frac{1}{4}x^{-6}}\,dx$$
$$=\frac{1}{2}\int_1^2(x^3+x^{-3})\,dx$$
$$=\frac{1}{2}\left[\frac{1}{4}x^4-\frac{1}{2x^2}\right]_1^2=\frac{33}{16}$$

(3) $y=\frac{1}{12}x^6+\frac{1}{8}x^{-4}$ $(1\le x\le 2)$

[Sol] $\frac{dy}{dx}=\frac{1}{2}x^5-\frac{1}{2}x^{-5}=\frac{1}{2}(x^5-x^{-5})$

Letting L be the length of the curve,

$$L=\int_1^2\sqrt{1+\left(\frac{1}{4}x^{10}-\frac{1}{2}+\frac{1}{4}x^{-10}\right)}\,dx=\int_1^2\sqrt{\frac{1}{4}x^{10}+\frac{1}{2}+\frac{1}{4}x^{-10}}\,dx$$
$$=\frac{1}{2}\int_1^2(x^5+x^{-5})\,dx$$
$$=\frac{1}{2}\left[\frac{x^6}{6}-\frac{1}{4x^4}\right]_1^2=\frac{687}{128}$$

O 162 a Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

In each exercise, determine the length of the given curve.

(1) $y=\frac{1}{2}x^2-\frac{1}{4}\ln x \qquad (1 \leq x \leq 2)$

[Sol] $\frac{dy}{dx}=x-\frac{1}{4x}$

Letting L be the length of the curve,

$$L=\int_1^2\sqrt{1+\left(x^2-\frac{1}{2}+\frac{1}{16x^2}\right)}dx=\int_1^2\sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}dx$$

$$=\int_1^2\left(x+\frac{1}{4x}\right)dx$$

$$=\left[\frac{x^2}{2}+\frac{1}{4}\ln x\right]_1^2$$

$$=\frac{3}{2}+\frac{1}{4}\ln 2$$

(2) $y=\ln(\cos x) \qquad \left(\frac{\pi}{4} \leq x \leq \frac{\pi}{3}\right)$

$\left(\textbf{Hint: } \int\frac{dx}{\cos x}=\frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right|+C\right)$

[Sol] $\frac{dy}{dx}=\frac{1}{\cos x}\cdot(-\sin x)=-\tan x$

Letting L be the length of the curve,

$$L=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sqrt{1+(-\tan x)^2}dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sqrt{\frac{1}{\cos^2 x}}dx$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{dx}{\cos x}$$

From the hint

$$=\left[\frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$=\frac{1}{2}\ln\left|\frac{\sqrt{3}+2}{\sqrt{3}-2}\right|-\frac{1}{2}\ln\left|\frac{1+\sqrt{2}}{1-\sqrt{2}}\right|=\ln\left|\left[(\sqrt{3}+2)^2(1-\sqrt{2})^2\right]^{\frac{1}{2}}\right|$$

$$=\ln\left|(\sqrt{3}+2)(1-\sqrt{2})\right|$$

O 162 b

(3) $y=\dfrac{e^x+e^{-x}}{2}\qquad (0\leq x\leq 2)$

[Sol] $\dfrac{dy}{dx}=\dfrac{1}{2}(e^x-e^{-x})$

Letting L be the length of the curve,

$$
\begin{aligned}
L &= \int_0^2 \sqrt{1+\left(\frac{1}{4}e^{2x}-\frac{1}{2}+\frac{1}{4}e^{-2x}\right)}\,dx \\
&= \int_0^2 \sqrt{\frac{1}{4}e^{2x}+\frac{1}{2}+\frac{1}{4}e^{-2x}}\,dx \\
&= \frac{1}{2}\int_0^2 (e^x+e^{-x})\,dx \\
&= \frac{1}{2}\Big[e^x-e^{-x}\Big]_0^2 \\
&= \frac{1}{2}\left(e^2-\frac{1}{e^2}\right) \\
&= \frac{e^4-1}{2e^2}
\end{aligned}
$$

$$\left[=\frac{e^2-e^{-2}}{2}\right]$$

KUMON

O 163 a Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

In each exercise, determine the length of the given curve.

(1) $y=\frac{1}{3}x^{\frac{3}{2}}-x^{\frac{1}{2}} \qquad (4\leq x\leq 16)$

[Sol] $\frac{dy}{dx}=\frac{1}{2}x^{\frac{1}{2}}-\frac{1}{2}x^{-\frac{1}{2}}$

Letting L be the length of the curve,

$$L=\int_4^{16}\sqrt{1+\frac{1}{4}x-\frac{1}{2}+\frac{1}{4x}}\,dx$$

$$=\frac{1}{2}\int_4^{16}\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx$$

$$=\frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}\right]_4^{16}$$

$$=\frac{62}{3}$$

(2) $y=\ln(\sin x) \qquad \left(\frac{\pi}{6}\leq x\leq\frac{3\pi}{4}\right)$

$\left(\textbf{Hint: }\int\frac{dx}{\sin x}=\frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+C\right)$

[Sol] $\frac{dy}{dx}=\frac{\cos x}{\sin x}$

Letting L be the length of the curve,

$$L=\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}}\sqrt{1+\frac{\cos^2 x}{\sin^2 x}}\,dx$$

$$=\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}}\frac{dx}{\sin x}$$

$$=\left[\frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|\right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}}$$

$$=\frac{1}{2}\ln\left|\frac{\sqrt{2}+2}{\sqrt{2}-2}\right|-\frac{1}{2}\ln\left|\frac{\sqrt{3}-2}{\sqrt{3}+2}\right|$$

$$=\ln\{(1+\sqrt{2})(\sqrt{3}+2)\}$$

O 163 b

(3) $y = x^2 \quad \left(0 \le x \le \frac{\sqrt{3}}{2}\right)$

[Sol] $\frac{dy}{dx} = 2x$

Letting L be the length of the curve,

$$L = \int_0^{\frac{\sqrt{3}}{2}} \sqrt{1 + 4x^2}\,dx$$

Letting $x = \frac{1}{2}\tan\theta, \quad dx = \frac{d\theta}{2\cos^2\theta} \quad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

When $x = 0, \quad \theta = 0$

When $x = \frac{\sqrt{3}}{2}, \quad \theta = \frac{\pi}{3}$

Substituting,

$$L = \int_0^{\frac{\pi}{3}} \left[\frac{1}{\cos\theta}\left(\frac{1}{2\cos^2\theta}\right)\right] d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos^3\theta}$$

Since $\int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos^3\theta} = \left[\frac{\tan\theta}{\cos\theta}\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \frac{\sin\theta\tan\theta}{\cos^2\theta}\,d\theta$

$$= \frac{\sqrt{3}}{\frac{1}{2}} - \int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^3\theta}\,d\theta$$

$$= 2\sqrt{3} - \int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos^3\theta} + \int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos\theta}$$

$$2\int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos^3\theta} = 2\sqrt{3} + \int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos\theta}$$

$$= 2\sqrt{3} + \left[\frac{1}{2}\ln\left|\frac{\sin\theta + 1}{\sin\theta - 1}\right|\right]_0^{\frac{\pi}{3}}$$

$$= 2\sqrt{3} + \frac{1}{2}\ln\left|\frac{\sqrt{3}+2}{\sqrt{3}-2}\right| - \frac{1}{2}\ln|-1|$$

$$= 2\sqrt{3} + \ln|\sqrt{3}+2|$$

$$\therefore L = \frac{\sqrt{3}}{2} + \frac{1}{4}\ln\left(\sqrt{3}+2\right)$$

Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

In each exercise, determine the length of the given curve.

(1) $y=\frac{3}{4}x^{\frac{4}{3}}-\frac{3}{8}x^{\frac{2}{3}}+12 \qquad (0\leq x\leq 8)$

[Sol] $\frac{dy}{dx}=x^{\frac{1}{3}}-\frac{1}{4}x^{-\frac{1}{3}}$

Letting L be the length of the curve,

$$L=\int_0^8\sqrt{1+x^{\frac{2}{3}}-\frac{1}{2}+\frac{1}{16}x^{-\frac{2}{3}}}\,dx$$

$$=\int_0^8\left(x^{\frac{1}{3}}+\frac{1}{4}x^{-\frac{1}{3}}\right)dx$$

$$=\left[\frac{3}{4}x^{\frac{4}{3}}+\frac{3}{8}x^{\frac{2}{3}}\right]_0^8=\frac{27}{2}$$

(2) $y=\frac{1}{4}x^2 \qquad (0\leq x\leq 2)$

[Sol] $\frac{dy}{dx}=\frac{1}{2}x$

Letting L be the length of the curve,

$$L=\int_0^2\sqrt{1+\frac{1}{4}x^2}\,dx$$

Letting $x=2\tan\theta$, $\quad dx=\frac{2}{\cos^2\theta}d\theta \quad \left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$

When $x=0$, $\theta=0$

When $x=2$, $\theta=\frac{\pi}{4}$

$$L=\int_0^{\frac{\pi}{4}}\left[\sqrt{1+\tan^2\theta}\left(\frac{2}{\cos^2\theta}\right)\right]d\theta=\int_0^{\frac{\pi}{4}}\frac{2}{\cos^3\theta}d\theta$$

Since $\displaystyle\int_0^{\frac{\pi}{4}}\frac{d\theta}{\cos^3\theta}=\left[\frac{\sin\theta}{\cos^2\theta}\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{d\theta}{\cos^3\theta}+\int_0^{\frac{\pi}{4}}\frac{d\theta}{\cos\theta}$

$$2\int_0^{\frac{\pi}{4}}\frac{d\theta}{\cos^3\theta}=\sqrt{2}+\left[\frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right|\right]_0^{\frac{\pi}{4}}$$

$$\therefore L=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$$

O 164 b

(3) $y = \ln x \qquad \left(1 \leq x \leq \sqrt{3}\right)$

[Sol] $\dfrac{dy}{dx} = \dfrac{1}{x}$

Letting L be the length of the curve,

$$L = \int_1^{\sqrt{3}} \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^{\sqrt{3}} \left(\frac{1}{x}\sqrt{x^2+1}\right) dx$$

Letting $x = \tan\theta$, $\qquad dx = \dfrac{d\theta}{\cos^2\theta} \qquad \left(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\right)$

When $\quad x = 1, \qquad \theta = \dfrac{\pi}{4}$

When $\quad x = \sqrt{3}, \qquad \theta = \dfrac{\pi}{3}$

Substituting,

$$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left[\frac{\sqrt{\tan^2\theta + 1}}{\tan\theta}\left(\frac{1}{\cos^2\theta}\right)\right] d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\theta}{\sin\theta\cos^2\theta}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos^2\theta}\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\sin\theta}{\cos^2\theta} + \frac{1}{\sin\theta}\right) d\theta$$

$$= \left[\frac{1}{\cos\theta} + \frac{1}{2}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left[2 + \frac{1}{2}\ln\left(\frac{1}{3}\right)\right] - \left(\sqrt{2} + \frac{1}{2}\ln\left|\frac{1-\sqrt{2}}{1+\sqrt{2}}\right|\right)$$

$$= 2 - \sqrt{2} + \ln\left(\frac{\sqrt{3}+\sqrt{6}}{3}\right)$$

Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	1-

Letting L be the length of the curve $x=g(t)$, $y=h(t)$ (where $a \leq t \leq b$),

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_a^b \sqrt{[g'(t)]^2+[h'(t)]^2}\,dt$$

In each exercise, determine the length of the given curve.

(1) $\begin{cases} x=-e^t \sin t \\ y=-e^t \cos t \end{cases}$ $(0 \leq t \leq 1)$

[Sol] $\dfrac{dx}{dt}=-e^t\sin t-e^t\cos t$, $\dfrac{dy}{dt}=-e^t\cos t+e^t\sin t$

Letting L be the length of the curve,

$$L=\int_0^1 \sqrt{e^{2t}(\sin t+\cos t)^2+e^{2t}(\cos t-\sin t)^2}\,dt$$

$$=\sqrt{2}\int_0^1 e^t\,dt$$

$$=\sqrt{2}\,(e-1)$$

(2) $\begin{cases} x=e^{2t}+e^{-2t} \\ y=4t \end{cases}$ $\left(\dfrac{1}{2} \leq t \leq 1\right)$

[Sol] $\dfrac{dx}{dt}=2e^{2t}-2e^{-2t}$, $\dfrac{dy}{dt}=4$

Letting L be the length of the curve,

$$L=\int_{\frac{1}{2}}^1 \sqrt{4(e^{2t}-e^{-2t})^2+16}\,dt$$

$$=2\int_{\frac{1}{2}}^1 (e^{2t}+e^{-2t})\,dt$$

$$=2\left[\frac{1}{2}e^{2t}-\frac{1}{2e^{2t}}\right]_{\frac{1}{2}}^1$$

$$=e^2-\frac{1}{e^2}-e+\frac{1}{e}=\frac{e^4-e^3+e-1}{e^2}$$

O 165 b

(3) $\begin{cases} x = 3t^2 + 4 \\ y = 3t^3 - 2 \end{cases} \qquad \left(\frac{1}{2} \le t \le 1\right)$

[Sol] $\dfrac{dx}{dt} = 6t, \qquad \dfrac{dy}{dt} = 9t^2$

Letting L be the length of the curve,

$$L = \int_{\frac{1}{2}}^{1} \sqrt{36t^2 + 81t^4}\, dt$$

$$= \int_{\frac{1}{2}}^{1} 3t\sqrt{4 + 9t^2}\, dt$$

Letting $u = 4 + 9t^2$, $\quad du = 18t\, dt$

When $\quad t = \dfrac{1}{2}, \qquad u = \dfrac{25}{4}$

When $\quad t = 1, \qquad u = 13$

$$\therefore L = \frac{1}{6}\int_{\frac{25}{4}}^{13} \sqrt{u}\, du$$

$$= \frac{1}{9}\left[u^{\frac{3}{2}}\right]_{\frac{25}{4}}^{13}$$

$$= \frac{1}{9}\left(13\sqrt{13} - \frac{125}{8}\right)$$

Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	–	–	–

In each exercise, determine the length of the given curve.

(1) $\begin{cases} x = e^t \\ y = e^t \end{cases}$ $(0 \leq t \leq 2)$

[Sol] $\dfrac{dx}{dt} = e^t, \quad \dfrac{dy}{dt} = e^t$

Letting L be the length of the curve,

$$L = \int_0^2 \sqrt{e^{2t} + e^{2t}}\,dt$$

$$= \sqrt{2}\int_0^2 e^t\,dt$$

$$= \sqrt{2}\Big[e^t\Big]_0^2$$

$$= \sqrt{2}(e^2 - 1)$$

(2) $\begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases}$ $(0 \leq \theta \leq \pi)$

[Sol] $\dfrac{dx}{d\theta} = r\cos\theta, \quad \dfrac{dy}{d\theta} = -r\sin\theta$

Letting L be the length of the curve,

$$L = \int_0^\pi \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\,d\theta$$

$$= \int_0^\pi r\,d\theta$$

$$= r\Big[\theta\Big]_0^\pi$$

$$= \pi r$$

O 166 b

(3) $\begin{cases} x = \sin^2 t \\ y = \cos^2 t \end{cases} \quad \left(\frac{\pi}{6} \le t \le \frac{3\pi}{2}\right)$

[Sol] $\frac{dx}{dt} = 2\sin t \cos t, \qquad \frac{dy}{dt} = -2\cos t \sin t$

Letting L be the length of the curve,

$$L = \int_{\frac{\pi}{6}}^{\frac{3\pi}{2}} \sqrt{4\sin^2 t\cos^2 t + 4\sin^2 t\cos^2 t}\,dt$$

$$= \int_{\frac{\pi}{6}}^{\frac{3\pi}{2}} (2\sqrt{2}\sin t\cos t)\,dt$$

Letting $u = \sin t, \qquad du = \cos t\,dt$

When $t = \frac{\pi}{6}, \qquad u = \frac{1}{2}$

When $t = \frac{3\pi}{2}, \qquad u = -1$

$$\therefore L = \int_{\frac{1}{2}}^{-1} 2\sqrt{2}u\,du$$

$$= \sqrt{2}\left[u^2\right]_{\frac{1}{2}}^{-1}$$

$$= \sqrt{2}\left(1 - \frac{1}{4}\right)$$

$$= \frac{3\sqrt{2}}{4}$$

KUMON

O 167 a

Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

In each exercise, determine the length of the given curve.

(1) $\begin{cases} x = 2t \\ y = e^t + e^{-t} \end{cases}$ $(1 \leq t \leq 4)$

[Sol] $\dfrac{dx}{dt} = 2,\quad \dfrac{dy}{dt} = e^t - e^{-t}$

Letting L be the length of the line,

$$L = \int_1^4 \sqrt{4 + e^{2t} - 2 + e^{-2t}}\,dt$$

$$= \int_1^4 (e^t + e^{-t})\,dt$$

$$= \Big[e^t - e^{-t}\Big]_1^4$$

$$= \frac{e^8 - e^5 + e^3 - 1}{e^4}$$

(2) $\begin{cases} x = \ln t \\ y = 1 + \ln t \end{cases}$ $(1 \leq t \leq 2)$

[Sol] $\dfrac{dx}{dt} = \dfrac{1}{t},\quad \dfrac{dy}{dt} = \dfrac{1}{t}$

Letting L be the length of the curve,

$$L = \int_1^2 \sqrt{\frac{1}{t^2} + \frac{1}{t^2}}\,dt$$

$$= \sqrt{2}\int_1^2 \frac{1}{t}\,dt$$

$$= \sqrt{2}\Big[\ln|t|\Big]_1^2$$

$$= \sqrt{2}\ln 2$$

O 167 b

(3) $\begin{cases} x = \sin t - \cos t \\ y = \sin t + \cos t \end{cases} \quad \left(0 \le t \le \dfrac{\pi}{2}\right)$

[Sol] $\dfrac{dx}{dt} = \cos t + \sin t, \quad \dfrac{dy}{dt} = \cos t - \sin t$

Letting L be the length of the curve,

$$L = \int_0^{\frac{\pi}{2}} \sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t + 2\cos t \sin t + \sin^2 t + \cos^2 t - 2\cos t \sin t + \sin^2 t}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{2}\, dt$$

$$= \sqrt{2}\Big[t\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\sqrt{2}\pi}{2}$$

(4) $\begin{cases} x = t^3 \\ y = \dfrac{3t^2}{2} \end{cases} \quad (1 \le t \le 8)$

[Sol] $\dfrac{dx}{dt} = 3t^2, \quad \dfrac{dy}{dt} = 3t$

Letting L be the length of the curve,

$$L = \int_1^8 \sqrt{9t^4 + 9t^2}\, dt$$

$$= \int_1^8 3t\sqrt{t^2 + 1}\, dt$$

$$= \Big[(t^2 + 1)^{\frac{3}{2}}\Big]_1^8$$

$$= 65^{\frac{3}{2}} - 2^{\frac{3}{2}}$$

$$\left[= 65\sqrt{65} - 2\sqrt{2}\right]$$

Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

In each exercise, determine the length of the given curve.

(1) $\begin{cases} x = t\cos t - \sin t \\ y = t\sin t + \cos t \end{cases}$ $(0 \leqq t \leqq 2\pi)$

[Sol] $\dfrac{dx}{dt} = \cos t - t\sin t - \cos t$

$= -t\sin t$

$\dfrac{dy}{dt} = \sin t + t\cos t - \sin t$

$= t\cos t$

Letting L be the length of the curve,

$$L = \int_0^{2\pi} \sqrt{t^2\sin^2 t + t^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} t\, dt$$

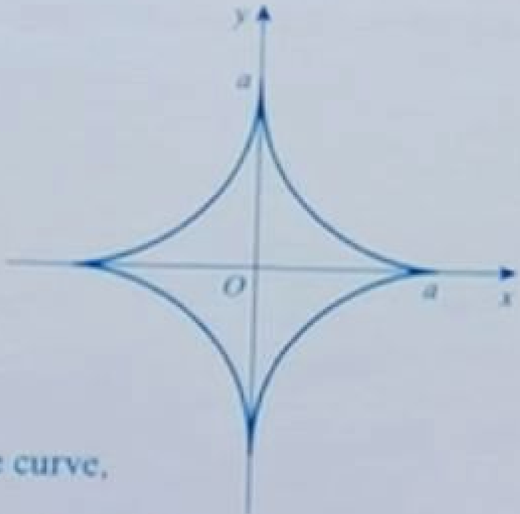
$$= \frac{1}{2}\Big[t^2\Big]_0^{2\pi}$$

$$= 2\pi^2$$

O 168 b

(2) $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $\quad (a>0,\ 0 \le t \le 2\pi)$

[Sol] The graph of the curve
is shown at the right.



Letting L be the length of the curve,
$\frac{1}{4}L$ is the length of the curve in the 1st Quadrant.

$$\frac{dx}{dt} = 3a\cos^2 t\,(-\sin t)$$

$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\therefore \frac{1}{4}L = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\sin^2 t\cos^2 t\,(\sin^2 t + \cos^2 t)}\,dt$$

$$= \int_0^{\frac{\pi}{2}} 3a\sin t\cos t\,dt$$

$$\therefore L = 4\int_0^{\frac{\pi}{2}} 3a\sin t\cos t\,dt$$

$$= 6a\int_0^{\frac{\pi}{2}} \sin 2t\,dt$$

$$= -3a\Big[\cos 2t\Big]_0^{\frac{\pi}{2}}$$

$$= 6a$$

Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	–	–	–	1–

1. In each exercise, determine the length of the given curve.

(1) $y=\frac{1}{3}(x^2+2)^{\frac{3}{2}}$ $\quad(-1\leq x\leq 2)$

[Sol] $\frac{dy}{dx}=x(x^2+2)^{\frac{1}{2}}$

Letting L be the length of the curve,

$$L=\int_{-1}^{2}\sqrt{1+x^2(x^2+2)}\,dx$$
$$=\int_{-1}^{2}(x^2+1)\,dx$$
$$=\left[\frac{1}{3}x^3+x\right]_{-1}^{2}$$
$$=6$$

(2) $\begin{cases}x=t-\sin t\\ y=1-\cos t\end{cases}$ $\quad(0\leq t\leq 2\pi)$

[Sol] $\frac{dx}{dt}=1-\cos t,\quad \frac{dy}{dt}=\sin t$

Letting L be the length of the curve,

$$L=\int_{0}^{2\pi}\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt$$
$$=\int_{0}^{2\pi}\sqrt{2-2\cos t}\,dt$$
$$=\int_{0}^{2\pi}\sqrt{2\left(2\sin^2\frac{t}{2}\right)}\,dt$$
$$=\int_{0}^{2\pi}2\sin\frac{t}{2}\,dt$$
$$=\left[-4\cos\frac{t}{2}\right]_{0}^{2\pi}$$
$$=-4(-1-1)$$
$$=8$$

O 169 b

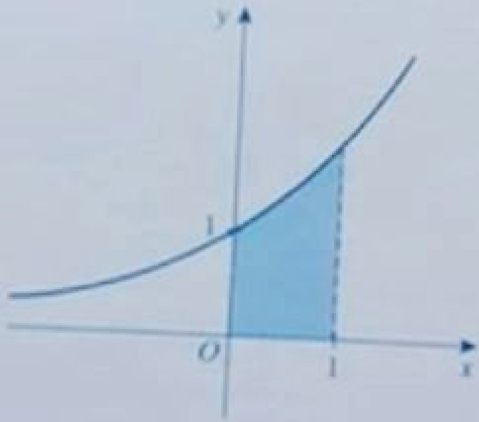
2. Determine the volume generated when the region bounded by the curve $y=a^x$ $(a>0)$ and the lines $x=0$ and $x=1$ is rotated around the x-axis.

(**Hint:** consider the different cases for a)

[Sol]

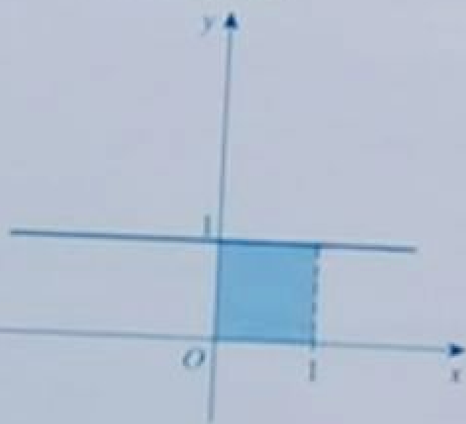
(a) When $a>1$, the graph of the curve is shown below.
Letting V be the volume,

$$V=\pi\int_0^1 (a^x)^2\,dx$$
$$=\pi\int_0^1 a^{2x}\,dx$$
$$=\pi\left[\frac{a^{2x}}{2\ln a}\right]_0^1$$
$$=\frac{\pi}{2\ln a}(a^2-1)$$



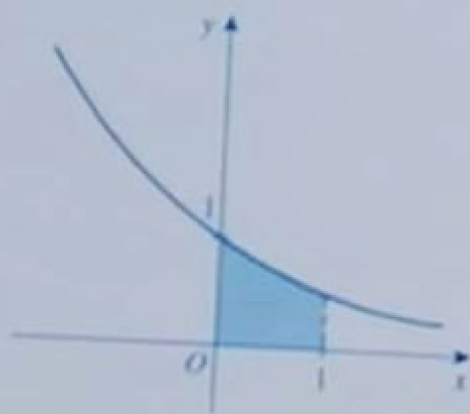
(b) When $a=1$, the graph of the curve is shown below.
Letting V be the volume,

$$V=\pi\int_0^1 (1^x)^2\,dx$$
$$=\pi\int_0^1 dx$$
$$=\pi\Big[x\Big]_0^1$$
$$=\pi$$



(c) When $a<1$, the graph of the curve is shown below.
Letting V be the volume,

$$V=\pi\int_0^1 (a^x)^2\,dx$$
$$=\pi\int_0^1 a^{2x}\,dx$$
$$=\pi\left[\frac{a^{2x}}{2\ln a}\right]_0^1$$
$$=\frac{\pi}{2\ln a}(a^2-1)$$



Applications of Integrals 4

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

1. The curve $y = e^{x-1}$ bounded by $x = 0$ and $x = 1$ forms a region. Determine the volume of the solid generated when the region is rotated around the x-axis.

[Sol] $V = \pi \int_0^1 (e^{x-1})^2 \, dx = \pi \int_0^1 e^{2x-2} \, dx$

$$= \frac{\pi}{2}\left[e^{2x-2}\right]_0^1$$

$$= \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)$$

$$= \frac{\pi(e^2-1)}{2e^2}$$

2. Determine the length of the given curve.

$$y = \frac{1}{4}x^2 - \frac{1}{2}\ln x \qquad (1 \leq x \leq 2)$$

[Sol] $\frac{dy}{dx} = \frac{1}{2}x - \frac{1}{2x}$

Letting L be the length of the curve,

$$L = \int_1^2 \sqrt{1 + \left(\frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}\right)} \, dx$$

$$= \int_1^2 \sqrt{\frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2}} \, dx$$

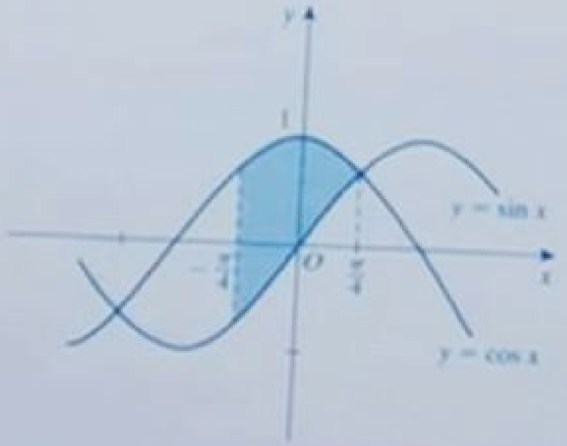
$$= \int_1^2 \left(\frac{1}{2}x + \frac{1}{2x}\right) dx$$

$$= \left[\frac{x^2}{4} + \frac{1}{2}\ln|x|\right]_1^2$$

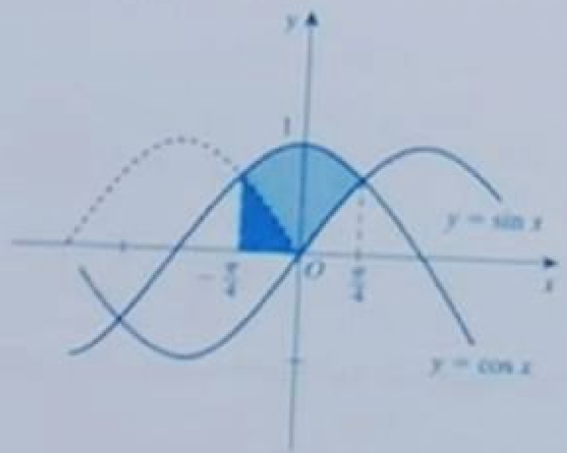
$$= \frac{3}{4} + \frac{1}{2}\ln 2$$

O 170 b

3. The two curves $y = \sin x$ and $y = \cos x$ bounded by $x = -\frac{\pi}{4}$ and by $x = \frac{\pi}{4}$ form a region. Determine the volume generated when the region is rotated around the x-axis.



[Sol] When the region is rotated around the x-axis there is an overlapping part. This overlapping part can be disregarded.



Letting V be the volume,

$$V = \pi\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x\, dx - \int_{0}^{\frac{\pi}{4}} \sin^2 x\, dx\right)$$

$$= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos 2x}{2}\, dx - \pi\int_{0}^{\frac{\pi}{4}} \frac{1-\cos 2x}{2}\, dx$$

$$= \pi\left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \pi\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_{0}^{\frac{\pi}{4}}$$

$$= \frac{1}{8}\pi(\pi + 6)$$

KUMON

O 171 a

Differential Equations 1

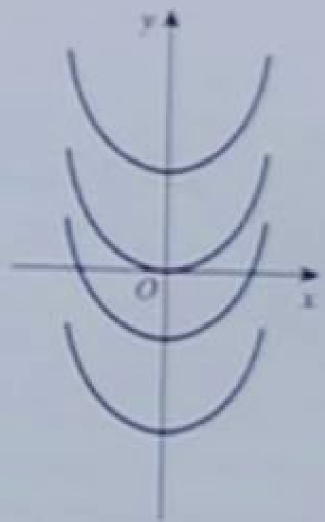
Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Given the curve $y = f(x)$ and the point $P(x, y)$ on it, assume the slope at P is always twice the value of the x coordinate of point P. Determine the equation of the curve.

[Sol] Since the slope at point P is $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \boxed{2x} \qquad \cdots①$$



Integrating both sides with respect to x,

$$y = \boxed{x^2} + C \qquad \cdots②$$

(where C is an arbitrary constant)

Thus, the equation of the curve is $y = \boxed{x^2} + C$.

Equations like ① that include derivatives of y (such as $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc.) are called *Differential Equations.*

Solving Differential Equations is the process of obtaining functions like ② from ①.

General Solutions are solutions like ② that contain an arbitrary constant.

Particular Solutions are solutions that are similar to ②, but do not contain an arbitrary constant. For example, if we let $x = 1$ and $y = 3$, then $C = 2$. Thus, the particular solution would be $y = x^2 + 2$.

2. Given the curve $y = f(x)$ and the point $K(x, y)$ on it, assume the slope at K is always 1 unit greater than the value of the x-coordinate of point K. Determine the equation of the curve.

[Sol] Since the slope at point K is $\frac{dy}{dx}$,

$$\frac{dy}{dx} = x + 1 \qquad \cdots①$$

Integrating both sides with respect to x,

$$y = \frac{1}{2}x^2 + x + C \qquad \cdots②$$

Thus, the equation of the curve is $y = \frac{1}{2}x^2 + x + C$.

3. Given the curve $y = f(x)$ and the point $M(x, y)$ on it, assume the slope at M can be expressed as the reciprocal of the value of the x-coordinate of point M. Determine the equation of the curve.

[Sol] Since the slope at point M is $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{1}{x} \qquad \cdots①$$

Integrating both sides with respect to x,

$$y = \ln x + C \qquad \cdots②$$

Thus, the equation of the curve is $y = \ln x + C$.

KUMON

Differential Equations 1

Time : to : Date Name

100%	90%	80%	70%	69%–

In each exercise, obtain a differential equation by eliminating the arbitrary constant c from the given equation.

Ex. $y = cx + 1$

[Sol] From the given, $y = cx + 1$ …①

Differentiating both sides of ① with respect to x,

$$\frac{dy}{dx} = c \quad \text{…②}$$

Eliminating c from ① and ②,

$$y = x\frac{dy}{dx} + 1$$

Thus, $$\frac{dy}{dx} = \frac{y-1}{x}$$

(1) $y = cx^2$ …①

[Sol] Differentiating both sides of ① with respect to x,

$$\frac{dy}{dx} = 2cx \quad \text{…②}$$

Eliminating c from ① and ②,

$$\frac{dy}{dx} = \frac{2y}{x}$$

(2) $y = 2cx - x + 1$ …①

[Sol] Differentiating both sides of ① with respect to x and rearranging,

$$c = \frac{1}{2}\left(\frac{dy}{dx} + 1\right) \quad \text{…②}$$

Eliminating c from ① and ②,

$$y = \left(\frac{dy}{dx} + 1\right)x - x + 1$$

$$y = x\frac{dy}{dx} + x - x + 1$$

Thus, $$\frac{dy}{dx} = \frac{y-1}{x}$$

(3) $y^2 = 4cx$ $\quad\cdots①$

[Sol] Differentiating both sides of ① with respect to x,

$$2y\frac{dy}{dx} = 4c \quad \cdots②$$

Eliminating c from ① and ②,

$$y^2 = 2xy\frac{dy}{dx}$$

Thus, $$\frac{dy}{dx} = \frac{y}{2x}$$

(4) $x^2 + y^2 = c^2$

[Sol] Differentiating both sides with respect to x,

$$2x + 2y\frac{dy}{dx} = 0$$

Thus, $$\frac{dy}{dx} = -\frac{x}{y}$$

(5) $(x-c)^2 + y^2 = 1$ $\quad\cdots①$

[Sol] Differentiating both sides of ① with respect to x,

$$2(x-c) + 2y\frac{dy}{dx} = 0 \quad \cdots②$$

Eliminating c from ① and ②,

$$(y^2)\left(\frac{dy}{dx}\right)^2 + y^2 = 1$$

Thus, $$\left(\frac{dy}{dx}\right)^2 = \frac{1-y^2}{y^2}$$

KUMON

O 173 a Differential Equations 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

In each exercise, obtain a differential equation by eliminating the arbitrary constant c from the given equation.

(1) $y = ce^x$ …①

[Sol] Differentiating both sides of ① with respect to x,

$$\frac{dy}{dx} = ce^x \quad \cdots②$$

Eliminating c from both ① and ②,

$$\frac{dy}{dx} = y$$

(2) $y = c(\cos x)$ …①

[Sol] Differentiating both sides of ① with respect to x,

$$\frac{dy}{dx} = -c\sin x \quad \cdots②$$

Eliminating c from both ① and ②,

$$y = \left(-\frac{dy}{dx}\cdot\frac{1}{\sin x}\right)\cdot\cos x = -\frac{dy}{dx}\cdot\frac{1}{\tan x}$$

Thus, $\dfrac{dy}{dx} = -y\tan x$

(3) $y = 3c^2e^{2x}$ …①

[Sol] Differentiating both sides of ① with respect to x,

$$\frac{dy}{dx} = 6c^2e^{2x} \quad \cdots②$$

Eliminating c from both ① and ②,

$$\frac{dy}{dx} = 2y$$

O 173 b

(4) $y = -\dfrac{\sin x}{c^2}$ …①

[Sol] Differentiating both sides of ① with respect to x,

$$\frac{dy}{dx} = -\frac{1}{c^2}\cdot\cos x \qquad \cdots②$$

Eliminating c from both ① and ②,

$$y = \left(\frac{dy}{dx}\cdot\frac{1}{\cos x}\right)\cdot\sin x = \frac{dy}{dx}\cdot\tan x$$

Thus, $\dfrac{dy}{dx} = \dfrac{y}{\tan x}$

(5) $y = -c(\tan x)$ …①

[Sol] Differentiating both sides of ① with respect to x,

$$\frac{dy}{dx} = -c\cdot\frac{1}{\cos^2 x} \qquad \cdots②$$

Eliminating c from both ① and ②,

$$y = (\cos^2 x)\cdot\frac{dy}{dx}\cdot\tan x$$

$$= \frac{dy}{dx}\cdot(\sin x)(\cos x)$$

Thus, $\dfrac{dy}{dx} = \dfrac{y}{\sin x\cos x}$

KUMON

O 174 a

Differential Equations 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

In each exercise, obtain a differential equation by eliminating the arbitrary constants A and B from the given equation.

(1) $y = Ax^2 + B$

[Sol] Differentiating both sides with respect to x,

$$\frac{dy}{dx} = \boxed{2Ax} \qquad \cdots①$$

Differentiating both sides again with respect to x,

$$\frac{d^2y}{dx^2} = \boxed{2A}$$

Therefore, $A = \boxed{\frac{1}{2}} \cdot \frac{d^2y}{dx^2} \qquad \cdots②$

Substituting ② into ①,

$$\frac{dy}{dx} = \boxed{x\frac{d^2y}{dx^2}}$$

(2) $y = \frac{1}{2}A + \frac{3}{4}Bx^2$

[Sol] Differentiating both sides with respect to x,

$$\frac{dy}{dx} = \frac{3}{2}Bx \qquad \cdots①$$

Differentiating both sides again with respect to x,

$$\frac{d^2y}{dx^2} = \frac{3}{2}B$$

Therefore, $B = \frac{2}{3} \cdot \frac{d^2y}{dx^2} \qquad \cdots②$

Substituting ② into ①,

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2}$$

O 174 b

(3) $y=(x-A)^2+B$

[Sol] Differentiating both sides with respect to x,

$$\frac{dy}{dx}=2(x-A)$$

Differentiating both sides again with respect to x,

$$\frac{d^2y}{dx^2}=2$$

(4) $y=(x-A)^2-(x-B)^2$

[Sol] Differentiating both sides with respect to x,

$$\frac{dy}{dx}=2(x-A)-2(x-B)$$

Differentiating both sides again with respect to x,

$$\frac{d^2y}{dx^2}=2-2=0$$

(5) $y=A\sin x+B\cos x-1$ …①

[Sol] Differentiating both sides of ① with respect to x,

$$\frac{dy}{dx}=A\cos x-B\sin x$$

Differentiating both sides again with respect to x,

$$\frac{d^2y}{dx^2}=-A\sin x-B\cos x \quad \cdots②$$

Eliminating A and B from ① and ②,

$$y=-\frac{d^2y}{dx^2}-1$$

Thus, $\dfrac{d^2y}{dx^2}=-1-y$

KUMON

O 175 a

Differential Equations 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

1. Show that each of the following functions is a solution to the differential equation $\left(\dfrac{dy}{dx}\right)^2 = 4y$.

(1) $y = x^2$

[Sol] Differentiating $y = x^2$ with respect to x,

$$\frac{dy}{dx} = \boxed{2x}$$

Squaring both sides,

$$\left(\frac{dy}{dx}\right)^2 = \boxed{4x^2}$$

Since $y = x^2$

$$= 4y$$

(2) $y = x^2 + 2x + 1$

[Sol] Differentiating $y = x^2 + 2x + 1$ with respect to x,

$$\frac{dy}{dx} = 2x + 2$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = (4x^2 + 8x + 4)$$

$$= 4(x^2 + 2x + 1)$$

$$= 4y$$

(3) $y = (x + c)^2$ (where c is a constant)

[Sol] Differentiating $y = (x + c)^2$ with respect to x,

$$\frac{dy}{dx} = 2(x + c)$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = 4(x + c)^2$$

$$= 4y$$

O 175 b

2. Show that each of the following functions is a solution to the differential equation $\left(\frac{dy}{dx}\right)^2 = 16y$.

(1) $y = 4x^2$

[Sol] Differentiating $y = 4x^2$ with respect to x,

$$\frac{dy}{dx} = 8x$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = 64x^2$$

$$= 16y$$

(2) $y = 4x^2 - 12x + 9$

[Sol] Differentiating $y = 4x^2 - 12x + 9$ with respect to x,

$$\frac{dy}{dx} = 8x - 12$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = 64x^2 - 192x + 144$$

$$= 16y$$

(3) $y = (2x + c)^2$ (where c is a constant)

[Sol] Differentiating $y = (2x + c)^2$ with respect to x,

$$\frac{dy}{dx} = 2(2x + c) \cdot 2$$

$$= 4(2x + c)$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = 16(2x + c)^2$$

$$= 16y$$

KUMON

O 176 a

Differential Equations 1

Time : to : Date Name

100%	90%	80%	70%	69%–

1. Show that each of the following functions is a solution to the differential equation $\dfrac{dy}{dx}+y=1+x$ ...①.

(1) $y=x$

[Sol] Differentiating $y=x$ with respect to x,

$$\frac{dy}{dx}=\boxed{1}$$

Substituting into the LHS of ①,

$$\frac{dy}{dx}+y=1+y$$

$$=1+x$$

(2) $y=x-5e^{-x}$

[Sol] Differentiating $y=x-5e^{-x}$ with respect to x,

$$\frac{dy}{dx}=1+5e^{-x}$$

Substituting into the LHS of ①,

$$\frac{dy}{dx}+y=1+5e^{-x}+y$$

$$=1+5e^{-x}+(x-5e^{-x})=1+x$$

(3) $y=x+ce^{-x}$ (where c is a constant)

[Sol] Differentiating $y=x+ce^{-x}$ with respect to x,

$$\frac{dy}{dx}=1-ce^{-x}$$

Substituting into the LHS of ①,

$$\frac{dy}{dx}+y=1-ce^{-x}+y$$

$$=1-ce^{-x}+(x+ce^{-x})=1+x$$

O 176 b

2. Show that each of the following functions is a solution to the differential equation $\frac{dy}{dx}-y=2(1-x)$ ...①.

(1) $y=2x$

[Sol] Differentiating $y=2x$ with respect to x,

$$\frac{dy}{dx}=2$$

Substituting into the LHS of ①,

$$\frac{dy}{dx}-y=2-y$$

$$=2-2x=2(1-x)$$

(2) $y=2x-5e^x$

[Sol] Differentiating $y=2x-5e^x$ with respect to x,

$$\frac{dy}{dx}=2-5e^x$$

Substituting into the LHS of ①,

$$\frac{dy}{dx}-y=2-5e^x-y$$

$$=2-5e^x-(2x-5e^x)=2-2x=2(1-x)$$

(3) $y=2x+ce^x$ (where c is a constant)

[Sol] Differentiating $y=2x+ce^x$ with respect to x,

$$\frac{dy}{dx}=2+ce^x$$

Substituting into the LHS of ①,

$$\frac{dy}{dx}-y=2+ce^x-y$$

$$=2+ce^x-(2x+ce^x)$$
$$=2-2x=2(1-x)$$

O 177 a　Differential Equations 1

Time　:　to　:　Date　　Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Solve each of the the following differential equations.

Ex. $\dfrac{dy}{dx}=x^2+x$

[Sol] Integrating both sides with respect to x,

$$y=\int(x^2+x)\,dx$$

$$\therefore y=\frac{x^3}{3}+\frac{x^2}{2}+C \qquad \text{(where } C \text{ is an arbitrary constant)}$$

(1) $\dfrac{dy}{dx}=x^3-2x$

[Sol] Integrating both sides with respect to x,

$$y=\int(x^3-2x)\,dx$$

$$\therefore y=\frac{x^4}{4}-x^2+C \qquad \text{(where } C \text{ is an arbitrary constant)}$$

(2) $\dfrac{dy}{dx}=\sqrt{x}+\sqrt[3]{x^2}$

[Sol] Integrating both sides with respect to x,

$$y=\int\left(\sqrt{x}+\sqrt[3]{x^2}\right)dx$$

$$\therefore y=\frac{2}{3}x^{\frac{3}{2}}+\frac{3}{5}x^{\frac{5}{3}}+C \qquad \text{(where } C \text{ is an arbitrary constant)}$$

$$\left[\therefore y=\frac{2}{3}x\sqrt{x}+\frac{3}{5}x\sqrt[3]{x^2}+C\right]$$

O 177 b

(3) $x^3\dfrac{d^2y}{dx^2}=1$

(**Hint:** More than one constant may be used.)

[Sol] Rearranging the original equation,

$$\frac{d^2y}{dx^2}=\boxed{\frac{1}{x^3}}$$

Integrating both sides with respect to x,

$$\frac{dy}{dx}=\int\frac{dx}{x^3}$$

$$=-\frac{1}{2x^2}+C_1$$

$$\therefore y=\int\left(-\frac{1}{2x^2}+C_1\right)dx$$

$$=\frac{1}{2x}+C_1x+C_2 \qquad \text{(where } C_1 \text{ and } C_2 \text{ are arbitrary constants)}$$

(4) $\sqrt{x}\dfrac{d^2y}{dx^2}=2$

[Sol] Rearranging the original equation,

$$\frac{d^2y}{dx^2}=\frac{2}{\sqrt{x}}$$

Integrating both sides with respect to x,

$$\frac{dy}{dx}=\int\frac{2}{\sqrt{x}}dx$$

$$=4\sqrt{x}+C_1$$

$$\therefore y=\int\left(4\sqrt{x}+C_1\right)dx$$

$$=\frac{8}{3}x^{\frac{3}{2}}+C_1x+C_2 \qquad \text{(where } C_1 \text{ and } C_2 \text{ are arbitrary constants)}$$

$$\left[y=\frac{8}{3}x\sqrt{x}+C_1x+C_2\right]$$

O 178 a Differential Equations 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(Mistakes) 0	-	1	-	2-

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx}=e^{2x}-3x+2$

[Sol] Integrating both sides with respect to x,

$$y=\int(e^{2x}-3x+2)\,dx$$
$$=\frac{1}{2}e^{2x}-\frac{3}{2}x^2+2x+C$$

(2) $\dfrac{dy}{dx}=\ln x+1$

[Sol] Integrating both sides with respect to x,

$$y=\int(\ln x+1)\,dx$$
$$=\int\ln x\,dx+\int dx$$
$$=(x\ln x-x)+x+C$$
$$=x\ln x+C$$

(3) $\dfrac{dy}{dx}=\ln x-\dfrac{1}{2}e^{-x}$

[Sol] Integrating both sides with respect to x,

$$y=\int\left(\ln x-\frac{1}{2}e^{-x}\right)dx$$
$$=\int\ln x\,dx-\frac{1}{2}\int e^{-x}\,dx$$
$$=x\ln x-x+\frac{1}{2e^{x}}+C$$

O 178 b

(4) $\dfrac{d^2y}{dx^2} = xe^x$

[Sol] Integrating both sides with respect to x,

$$\frac{dy}{dx} = \int xe^x dx$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + C_1$$
$$\therefore y = \int (xe^x - e^x + C_1) dx$$
$$= \int xe^x dx - \int e^x dx + \int C_1 dx$$
$$= (xe^x - e^x) - e^x + C_1 x + C_2$$
$$= xe^x - 2e^x + C_1 x + C_2$$

(5) $\dfrac{d^2y}{dx^2} = x\cos x$

[Sol] Integrating both sides with respect to x,

$$\frac{dy}{dx} = \int x\cos x\, dx$$
$$= x\sin x - \int \sin x\, dx$$
$$= x\sin x + \cos x + C_1$$
$$\therefore y = \int (x\sin x + \cos x + C_1) dx$$
$$= \int x\sin x\, dx + \int \cos x\, dx + \int C_1 dx$$
$$= \left(-x\cos x + \int \cos x\, dx\right) + \sin x + C_1 x + C_2$$
$$= -x\cos x + 2\sin x + C_1 x + C_2$$

O 179 a Differential Equations 1

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	1	–	2–

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx}=\sin x+\cos x$

[Sol] Integrating both sides with respect to x,

$$y=\int(\sin x+\cos x)\,dx$$
$$=-\cos x+\sin x+C$$

(2) $\dfrac{dy}{dx}=\dfrac{3\cos x}{\sin^2 x}$

[Sol] Integrating both sides with respect to x,

$$y=\int\frac{3\cos x}{\sin^2 x}\,dx$$

Letting $u=\sin x$,

$$y=\int\frac{3}{u^2}\,du$$
$$=-\frac{3}{u}+C=-\frac{3}{\sin x}+C$$

(3) $\dfrac{dy}{dx}=\sin x\cos x$

[Sol] Integrating both sides with respect to x,

$$y=\int\sin x\cos x\,dx$$

Letting $u=\sin x$,

$$y=\int u\,du$$
$$=\frac{u^2}{2}+C=\frac{1}{2}\sin^2 x+C$$

O 179 b

(4) $\frac{dy}{dx}=\frac{x}{(x^2+4)^2}$

[Sol] Integrating both sides with respect to x,

$$y=\int\frac{x}{(x^2+4)^2}dx$$

Letting $u=x^2+4$,

$$y=\int\frac{du}{2u^2}$$

$$=-\frac{1}{2u}+C$$

$$=-\frac{1}{2(x^2+4)}+C$$

(5) $\frac{d^2y}{dx^2}=\ln x$

[Sol] Integrating both sides with respect to x,

$$\frac{dy}{dx}=\int\ln x\,dx$$

$$=x\ln x-x+C_1$$

$$\therefore y=\int(x\ln x-x+C_1)\,dx$$

$$=\int x\ln x\,dx-\int x\,dx+\int C_1\,dx$$

$$=\left(\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\,dx\right)-\frac{x^2}{2}+C_1x+C_2$$

$$=\frac{1}{2}x^2\ln x-\frac{3}{4}x^2+C_1x+C_2$$

KUMON

O 180 a

Differential Equations 1

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

1. Obtain a differential equation by eliminating the arbitrary constant c from the given equation.

$y = -c\sin x$ $\quad\cdots①$

[Sol] Differentiating both sides of ① with respect to x,

$$\frac{dy}{dx} = -c(\cos x) \quad\cdots②$$

Eliminating c from both ① and ②,

$$y = \left(\frac{dy}{dx}\cdot\frac{1}{\cos x}\right)\cdot\sin x$$

$$= \frac{dy}{dx}\cdot\tan x$$

Thus, $\dfrac{dy}{dx} = \dfrac{y}{\tan x}$

2. Show that each of the following functions is a solution to the differential equation $\left(\dfrac{dy}{dx}\right)^2 = y$.

(1) $y = \dfrac{1}{4}x^2$

[Sol] Differentiating $y = \dfrac{1}{4}x^2$ with respect to x,

$$\frac{dy}{dx} = \frac{1}{2}x$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x^2 = y$$

(2) $y = \left(\dfrac{1}{2}x - 1\right)^2$

[Sol] Differentiating $y = \left(\dfrac{1}{2}x - 1\right)^2$ with respect to x,

$$\frac{dy}{dx} = 2\left(\frac{1}{2}x - 1\right)\cdot\frac{1}{2} = \frac{1}{2}x - 1$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x - 1\right)^2 = y$$

O 180 b

3. Solve each of the following differential equations.

(1) $\dfrac{dy}{dx}=2x^2-x$

[Sol] Integrating both sides with respect to x,

$$y=\int(2x^2-x)\,dx$$
$$=\frac{2}{3}x^3-\frac{1}{2}x^2+C$$

(2) $\dfrac{dy}{dx}=-x\sin x$

[Sol] Integrating both sides with respect to x,

$$y=\int -x\sin x\,dx$$
$$=x\cos x-\int\cos x\,dx$$
$$=x\cos x-\sin x+C$$

(3) $x^3\dfrac{d^2y}{dx^2}=-1$

[Sol] Rearranging the original equation,

$$\frac{d^2y}{dx^2}=-\frac{1}{x^3}$$

Integrating both sides with respect to x,

$$\frac{dy}{dx}=\int-\frac{dx}{x^3}=\frac{1}{2x^2}+C_1$$
$$\therefore y=\int\left(\frac{1}{2x^2}+C_1\right)dx$$
$$=-\frac{1}{2x}+C_1x+C_2$$

Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69% -
(mistakes) 0	-	-	-	-

To solve the differential equation $f(y)\dfrac{dy}{dx}=g(x)$ ...①,
we can use the method of separating variables.
Since y is a function of x, both sides of ① are functions of x.
Therefore, we can integrate both sides with respect to x.

$$\int f(y)\frac{dy}{dx}dx=\int g(x)dx$$

Since the left side can be written as $\int f(y)dy$,

$$\int f(y)dy=\int g(x)dx$$

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx}=\dfrac{2}{y}$

[Sol] Rearranging the original equation, $y\,dy=\boxed{2}\,dx$

Integrating both sides,

$$\int y\,dy=\int\boxed{2}\,dx$$

Therefore, $\boxed{\frac{1}{2}y^2}=\boxed{2x}+c_1$

$$\therefore y^2=\boxed{4}x+2c_1$$

Replacing $2c_1$ with c,

$y^2=\boxed{4x+c}$ (where c is an arbitrary constant)

O 181 b

(2) $\dfrac{dy}{dx}=\dfrac{3}{\sqrt{y}}$

[Sol] Rearranging the original equation,

$$\sqrt{y}\,dy=3\,dx$$

Integrating both sides,

$$\int\sqrt{y}\,dy=\int 3\,dx$$

Therefore, $\dfrac{2}{3}y^{\frac{3}{2}}=3x+c_1$

$$y^{\frac{3}{2}}=\frac{9}{2}x+\frac{3}{2}c_1$$

Replacing $\dfrac{3}{2}c_1$ with c,

$$y^{\frac{3}{2}}=\frac{9}{2}x+c$$

(3) $\dfrac{dy}{dx}=\dfrac{x}{y}$

[Sol] Rearranging the original equation,

$$y\,dy=x\,dx$$

Integrating both sides,

$$\int y\,dy=\int x\,dx$$

Therefore, $\dfrac{1}{2}y^2=\dfrac{1}{2}x^2+c_1$

$$y^2=x^2+2c_1$$

Replacing $2c_1$ with c,

$$y^2=x^2+c$$

Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx} = -y$ …①

[Sol] (a) When $y=0$, $\dfrac{dy}{dx}=0$. This satisfies ①, so $y=0$ is a solution.

(b) When $y \neq 0$, ① can be rewritten as:

$$\frac{1}{y}dy = \boxed{-1}\,dx$$

Integrating both sides,

$$\int \frac{1}{y}dy = \int (-1)\,dx$$

$$\therefore \boxed{\ln|y|} = -x + c$$

$$\therefore y = \pm e^{\boxed{-x+c}}$$

$$= \pm e^{c} \cdot \boxed{e^{-x}}$$

Replacing $\pm e^{c}$ with k,

$y = \boxed{ke^{-x}}$ (where k is an arbitrary constant, $k \neq 0$)

Note: The arbitrary constant can be any letter.

However, when $k=0$, $y=0$. This satisfies (a).

Therefore, the general solution is:

$y = \boxed{ke^{-x}}$ (where k is an arbitrary constant)

O 182 b

(2) $\dfrac{dy}{dx}=4xy$ $\qquad\cdots$①

[Sol] (a) When $y=0$, $\dfrac{dy}{dx}=0$. This satisfies ①, so $y=0$ is a solution.

(b) When $y\neq 0$, ① can be rewritten as:

$$\frac{dy}{y}=4x\,dx$$

Integrating both sides,

$$\int\frac{dy}{y}=\int 4x\,dx$$

$$\ln|y|=2x^2+c$$

Therefore, $y=\pm e^{2x^2+c}=\pm e^c\cdot e^{2x^2}$

Replacing $\pm e^c$ with k,

$y=ke^{2x^2}$ (where k is an arbitrary constant, $k\neq 0$)

However, when $k=0$, $y=0$. This satisfies (a).

Thus, the general solution is $y=ke^{2x^2}$, where k is an arbitrary constant.

(3) $\dfrac{dy}{dx}=6x^2y$ $\qquad\cdots$①

[Sol] (a) When $y=0$, $\dfrac{dy}{dx}=0$. This satisfies ①, so $y=0$ is a solution.

(b) When $y\neq 0$, ① can be rewritten as:

$$\frac{dy}{y}=6x^2dx$$

Integrating both sides,

$$\int\frac{dy}{y}=\int 6x^2dx$$

$$\ln|y|=2x^3+c$$

Therefore, $y=\pm e^{2x^3+c}=\pm e^c\cdot e^{2x^3}$

Replacing $\pm e^c$ with k,

$y=ke^{2x^3}$ (where k is an arbitrary constant, $k\neq 0$)

However, when $k=0$, $y=0$. This satisfies (a).

Thus, the general solution is $y=ke^{2x^3}$, where k is an arbitrary constant.

Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx}+y=1$ …①

[Sol] (a) When $y=1$, $\dfrac{dy}{dx}=\boxed{0}$. This satisfies ①, so $y=\boxed{1}$ is a solution.

(b) When $y\neq 1$, ① can be rewritten as:

$$\frac{dy}{1-y}=\boxed{dx}$$

Integrating both sides,

$$-\ln|1-y|=x+c$$

Therefore, $1-y=\pm e^{-(x+c)}=\pm e^{-c}\cdot e^{-x}$

Replacing $\pm e^{-c}$ with k,

$y=1-ke^{-x}$ (where k is an arbitrary constant, $k\neq 0$)

However, when $k=0$, $y=1$. This satisfies (a).

Thus, the general solution is $y=1-ke^{-x}$, where k is an arbitrary constant.

(2) $\dfrac{dy}{dx}=2y-5$ …①

[Sol] (a) When $y=\dfrac{5}{2}$, $\dfrac{dy}{dx}=0$. This satisfies ①, so $y=\dfrac{5}{2}$ is a solution.

(b) When $y\neq\dfrac{5}{2}$, ① can be rewritten as:

$$\frac{dy}{2y-5}=dx$$

Integrating both sides,

$$\frac{1}{2}\ln|2y-5|=x+c$$

Therefore, $2y-5=\pm e^{2(x+c)}=\pm e^{2c}\cdot e^{2x}$

Replacing $\pm e^{2c}$ with k_1, $y=\dfrac{5}{2}+\dfrac{1}{2}k_1e^{2x}$ $(k_1\neq 0)$

Replacing $\dfrac{1}{2}k_1$ with k, $y=\dfrac{5}{2}+ke^{2x}$ $(k\neq 0)$

However, when $k=0$, $y=\dfrac{5}{2}$. This satisfies (a).

Thus, the general solution is $y=\dfrac{5}{2}+ke^{2x}$, where k is an arbitrary constant.

O 183 b

(3) $x\dfrac{dy}{dx}+y=\dfrac{dy}{dx}+1$ $\quad\cdots$① $\left(\textbf{Hint: } \text{Solve for } \dfrac{dy}{dx}.\right)$

[Sol] (a) When $y=1$, $\dfrac{dy}{dx}=0$. This satisfies ①, so $y=1$ is a solution.

(b) When $y\neq 1$, ① can be rewritten as:

$$\frac{dy}{y-1}=-\frac{dx}{x-1}$$

Integrating both sides,

$$\int\frac{dy}{y-1}=-\int\frac{dx}{x-1}$$

$$\ln|y-1|=-\ln|x-1|+c$$

Therefore, $|y-1|=\dfrac{1}{|x-1|}\cdot e^c$

$$y=1\pm\frac{e^c}{x-1}$$

Replacing $\pm e^c$ with k,

$$y=1+\frac{k}{x-1}\quad(\text{where } k \text{ is an arbitrary constant, } k\neq 0)$$

However, when $k=0$, $y=1$. This satisfies (a).

Thus, the general solution is $y=1+\dfrac{k}{x-1}$, where k is an arbitrary constant.

(4) $y+\dfrac{dy}{dx}=2x\dfrac{dy}{dx}$ $\quad\cdots$①

[Sol] (a) When $y=0$, $\dfrac{dy}{dx}=0$. This satisfies ①, so $y=0$ is a solution.

(b) When $y\neq 0$, ① can be rewritten as:

$$\frac{dy}{y}=\frac{dx}{2x-1}$$

Integrating both sides,

$$\int\frac{dy}{y}=\int\frac{dx}{2x-1}$$

$$\ln|y|=\frac{1}{2}\ln|2x-1|+c$$

$$2\ln|y|=\ln|2x-1|+2c$$

Therefore, $y^2=\pm(2x-1)\cdot e^{2c}$

Replacing $\pm e^{2c}$ with k,

$$y^2=k(2x-1)\quad(\text{where } k \text{ is an arbitrary constant, } k\neq 0)$$

However, when $k=0$, $y=0$. This satisfies (a).

Thus, the general solution is $y^2=k(2x-1)$, where k is an arbitrary constant.

KUMON

O 184 a

Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	-	-	1	2~

Solve each of the following differential equations.

(1) $x^2 \frac{dy}{dx} + 1 = 0$

[Sol] Rearranging the original equation,

$$dy = -\frac{dx}{x^2}$$

Integrating both sides,

$$\int dy = -\int \frac{dx}{x^2}$$

$$\therefore y = \frac{1}{x} + c \quad \text{(where } c \text{ is an arbitrary constant)}$$

(2) $(1 + x^2)y\frac{dy}{dx} + (1 + y^2)x = 0$ (**Hint:** Find the condition of the constant.)

[Sol] Rearranging the original equation,

$$\frac{y}{1+y^2}dy = -\frac{x}{1+x^2}dx$$

Integrating both sides,

$$\int \frac{y}{1+y^2}dy = -\int \frac{x}{1+x^2}dx$$

$$\frac{1}{2}\ln(1+y^2) = -\frac{1}{2}\ln(1+x^2) + c$$

$$\ln[(1+x^2)(1+y^2)] = 2c$$

$$(1+x^2)(1+y^2) = e^{2c}$$

Replacing e^{2c} with k,

$$(1+x^2)(1+y^2) = k \quad \text{(where } k \text{ is an arbitrary constant, } k > 0\text{)}$$

O 184 b

(3) $e^x \dfrac{dy}{dx} = x^2 + 1$

[Sol] Rearranging the original equation,

$$dy = \frac{x^2+1}{e^x} dx$$

Integrating both sides,

$$\int dy = \int \frac{x^2+1}{e^x} dx$$

$$y = -e^{-x}(x^2+1) + 2\int xe^{-x} dx$$

$$y = -e^{-x}(x^2+1) + 2\left(-xe^{-x} + \int e^{-x} dx\right)$$

$$y = -e^{-x}(x^2+1) + 2(-xe^{-x} - e^{-x} + c_1)$$

Replacing $2c_1$ with c,

$$y = -e^{-x}(x^2+2x+3) + c \quad \text{(where } c \text{ is an arbitrary constant)}$$

(4) $\dfrac{dy}{dx} = \dfrac{y-1}{2x+1}$ …①

[Sol] (a) When $y = 1$, $\dfrac{dy}{dx} = 0$. This satisfies ①, so $y = 1$ is a solution.

(b) When $y \neq 1$, ① can be rewritten as:

$$\frac{dy}{y-1} = \frac{dx}{2x+1}$$

Integrating both sides,

$$\int \frac{dy}{y-1} = \int \frac{dx}{2x+1}$$

$$\ln|y-1| = \frac{1}{2}\ln|2x+1| + c$$

$$\ln \frac{(y-1)^2}{|2x+1|} = 2c$$

$$\frac{(y-1)^2}{2x+1} = \pm e^{2c}$$

Replacing $\pm e^{2c}$ with k,

$$(y-1)^2 = k(2x+1) \quad \text{(where } k \text{ is an arbitrary constant, } k \neq 0)$$

However, when $k = 0$, $y = 1$. This satisfies (a).

Thus, the general solution is $(y-1)^2 = k(2x+1)$, where k is an arbitrary constant.

Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx}=e^{y}$

[Sol] Given $\dfrac{dy}{dx}=e^{y}$.

Since $e^{y}\neq 0$, we can take the reciprocal of each side.

$$\frac{dx}{dy}=\frac{1}{\dfrac{dy}{dx}}$$

$$\frac{dx}{dy}=e^{\boxed{-y}}$$

$$dx=e^{\boxed{-y}}dy$$

Integrating both sides,

$$\int dx=\int e^{\boxed{-y}}dy$$

$$\therefore\ x=\boxed{-e^{-y}+c}$$

(2) $\dfrac{dy}{dx}=-2e^{2y+3}$

[Sol] Given $\dfrac{dy}{dx}=-2e^{2y+3}$.

Since $e^{2y+3}\neq 0$, we can take the reciprocal of each side.

$$\frac{dx}{dy}=-\frac{1}{2}e^{-2y-3}$$

$$dx=-\frac{1}{2}e^{-2y-3}dy$$

Integrating both sides,

$$\int dx=-\frac{1}{2}\int e^{-2y-3}dy$$

$$\therefore\ x=\frac{1}{4}e^{-2y-3}+c$$

O 185 b

(3) $y\dfrac{dy}{dx}=\sqrt{1-y^2}$

[Sol] Given $y\dfrac{dy}{dx}=\sqrt{1-y^2}$ …①

(a) When $y=\pm1$, $\dfrac{dy}{dx}=0$. This satisfies ①, so $y=\pm1$ is a solution.

(b) When $y\neq\pm1$, ① can be rewritten as: $\dfrac{dy}{dx}=\boxed{\dfrac{\sqrt{1-y^2}}{y}}$

Since $\sqrt{1-y^2}\neq0$, we can take the reciprocal of each side.

$$\frac{dx}{dy}=\boxed{\frac{y}{\sqrt{1-y^2}}}$$

$$\therefore dx=\boxed{\frac{y}{\sqrt{1-y^2}}}\,dy$$

Integrating both sides,

$$\int dx=\int\frac{y}{\sqrt{1-y^2}}dy$$

$$\therefore x=-\sqrt{1-y^2}+c$$

Therefore, the solutions are: $y=\boxed{\pm1}$ and $x=\boxed{-\sqrt{1-y^2}+c}$

(4) $\cos y\dfrac{dy}{dx}=\sin^2 y$ $(0\leq y<\pi)$

[Sol] Given $\cos y\dfrac{dy}{dx}=\sin^2 y$ …①

(a) When $y=0$, $\dfrac{dy}{dx}=0$. This satisfies ①, so $y=0$ is a solution.

(b) When $y\neq0$, ① can be rewritten as: $\dfrac{dy}{dx}=\dfrac{\sin^2 y}{\cos y}$

Since $\sin^2 y\neq0$, we can take the reciprocal of each side.

$$\frac{dx}{dy}=\frac{\cos y}{\sin^2 y}$$

$$dx=\frac{\cos y}{\sin^2 y}dy$$

Integrating both sides,

$$\int dx=\int\frac{\cos y}{\sin^2 y}dy$$

$$\therefore x=-\frac{1}{\sin y}+c$$

Therefore, the solutions are: $y=0$ and $x=-\dfrac{1}{\sin y}+c$

KUMON

O 186 a

Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69%~
(mistakes) 0	–	–	1	2~

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx} = (y+1)^2$

[Sol] Given $\dfrac{dy}{dx} = (y+1)^2$ ...①

(a) When $y = -1$, $\dfrac{dy}{dx} = 0$. This satisfies ①, so $y = -1$ is a solution.

(b) When $y \neq -1$,

Since $(y+1)^2 \neq 0$, we can take the reciprocal of each side of ①.

$$\frac{dx}{dy} = \frac{1}{(y+1)^2}$$

$$dx = \frac{dy}{(y+1)^2}$$

Integrating both sides,

$$\int dx = \int \frac{dy}{(y+1)^2}$$

$$x = -\frac{1}{y+1} + c$$

Therefore, the solutions are: $y = -1$ and $x = -\dfrac{1}{y+1} + c$

(2) $\dfrac{dy}{dx} = (2-3y)^3$

[Sol] Given $\dfrac{dy}{dx} = (2-3y)^3$ ...①

(a) When $y = \dfrac{2}{3}$, $\dfrac{dy}{dx} = 0$. This satisfies ①, so $y = \dfrac{2}{3}$ is a solution.

(b) When $y \neq \dfrac{2}{3}$,

Since $(2-3y)^3 \neq 0$, we can take the reciprocal of each side of ①.

$$\frac{dx}{dy} = \frac{1}{(2-3y)^3}$$

$$dx = \frac{dy}{(2-3y)^3}$$

Integrating both sides,

$$\int dx = \int \frac{dy}{(2-3y)^3}$$

$$x = \frac{1}{6(2-3y)^2} + c$$

Therefore, the solutions are: $y = \dfrac{2}{3}$ and $x = \dfrac{1}{6(2-3y)^2} + c$

O 186 b

(3) $y\dfrac{dy}{dx} = 2y^2 + 1$

[Sol] Rearranging the original equation, $\dfrac{dy}{dx} = \dfrac{2y^2+1}{y}$ …①

Since $(2y^2 + 1) \neq 0$, we can take the reciprocals of each side of ①.

$$\frac{dx}{dy} = \frac{y}{2y^2+1}$$

$$dx = \frac{y}{2y^2+1}dy$$

Integrating both sides,

$$\int dx = \int \frac{y}{2y^2+1}dy$$

$$x = \frac{1}{4}\ln(2y^2+1) + c$$

$$2y^2 + 1 = e^{4x-4c}$$

Replacing e^{-4c} with k,

$$2y^2 = ke^{4x} - 1, \qquad \text{where } k > 0$$

(4) $8y^3\dfrac{dy}{dx} = y^4 + 3$

[Sol] Rearranging the original equation, $\dfrac{dy}{dx} = \dfrac{y^4+3}{8y^3}$ …①

Since $(y^4 + 3) \neq 0$, we can take the reciprocals of each side of ①.

$$\frac{dx}{dy} = \frac{8y^3}{y^4+3}$$

$$dx = \frac{8y^3}{y^4+3}dy$$

Integrating both sides,

$$\int dx = \int \frac{8y^3}{y^4+3}dy$$

$$x = 2\ln(y^4+3) + c$$

$$e^{\frac{1}{2}x-\frac{1}{2}c} = y^4 + 3$$

Replacing $e^{-\frac{1}{2}c}$ with k,

$$y^4 = ke^{\frac{1}{2}x} - 3, \qquad \text{where } k > 0$$

O 187 a

Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	1	2–

In each exercise, using the initial condition in the parentheses, solve each of the following differential equations.

(1) $(x-1)\dfrac{dy}{dx}=y$ $(x=0,\ y=1)$

[Sol] Since $x \neq 1$ and $y \neq 0$, we can arrange the original equation as:

$$\frac{1}{y}dy=\boxed{\frac{1}{x-1}}\,dx$$

Integrating both sides,

$$\int\frac{1}{y}dy=\int\boxed{\frac{1}{x-1}}\,dx$$

$$\therefore \ln|y|=\boxed{\ln|x-1|}+c$$

$$y=\boxed{(x-1)}\cdot\pm e^{c}$$

Replacing $\pm e^{c}$ with k,

$$y=\boxed{k(x-1)}$$

From the initial condition,

$$k=\boxed{-1}$$

Therefore, $y=\boxed{1-x}$

(2) $(x+3)\dfrac{dy}{dx}=y-2$ $(x=-1,\ y=1)$

[Sol] Since $x \neq -3$ and $y \neq 2$, we can arrange the original equation as:

$$\frac{dy}{y-2}=\frac{dx}{x+3}$$

Integrating both sides,

$$\int\frac{dy}{y-2}=\int\frac{dx}{x+3}$$

$$\ln|y-2|=\ln|x+3|+c$$

$$y-2=(x+3)\cdot\pm e^{c}$$

Replacing $\pm e^{c}$ with k, $y=2+k(x+3)$

From the initial condition, $k=-\dfrac{1}{2}$

Therefore, $y=\dfrac{1}{2}(1-x)$

O 187 b

(3) $\dfrac{dy}{dx} = 3y + 1$ $\qquad\qquad (x = 1,\ y = 1)$

[Sol] Since $y \neq -\dfrac{1}{3}$, we can arrange the original equation as:

$$\frac{dy}{3y+1} = dx$$

Integrating both sides,

$$\int \frac{dy}{3y+1} = \int dx$$

$$\frac{1}{3}\ln|3y+1| = x + c$$

$$3y + 1 = \pm e^{3x} \cdot e^{3c}$$

Replacing $\pm e^{3c}$ with k,

$$3y + 1 = ke^{3x}$$

From the initial condition,

$$k = 4e^{-3}$$

Therefore, $\quad 3y + 1 = 4e^{3x-3} \qquad \left[y = \dfrac{1}{3}(4e^{3x-3} - 1) \right]$

(4) $\sqrt{1+x}\,\dfrac{dy}{dx} = \sqrt{1+y}$ $\qquad\qquad (x = 3,\ y = 3)$

[Sol] Since $x \neq -1$ and $y \neq -1$, we can arrange the original equation as:

$$\frac{dy}{\sqrt{1+y}} = \frac{dx}{\sqrt{1+x}}$$

Integrating both sides,

$$\int \frac{dy}{\sqrt{1+y}} = \int \frac{dx}{\sqrt{1+x}}$$

$$2\sqrt{1+y} = 2\sqrt{1+x} + c$$

$$\sqrt{1+y} = \sqrt{1+x} + \frac{c}{2}$$

From the initial condition,

$$c = 0$$

Therefore, $\quad \sqrt{1+y} = \sqrt{1+x}$

KUMON

O 188 a Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

In each exercise, using the initial condition in the parentheses, solve each of the following differential equations.

(1) $y - x\dfrac{dy}{dx} = 1$ $\quad(x = 1,\ y = 2)$

[Sol] Since $x \neq 0$ and $y \neq 1$, we can arrange the original equation as:

$$\frac{dy}{y-1} = \frac{dx}{x}$$

Integrating both sides,

$$\int \frac{dy}{y-1} = \int \frac{dx}{x}$$

$$\ln|y-1| = \ln|x| + c$$

$$y - 1 = \pm x \cdot e^c$$

Replacing $\pm e^c$ with k,

$$y = 1 + kx$$

From the initial condition,

$$k = 1$$

Therefore, $\quad y = 1 + x$

(2) $\dfrac{dy}{dx} = e^y$ $\quad(x = 0,\ y = 0)$

[Sol] Rearranging the original equation,

$$e^{-y}\,dy = dx$$

Integrating both sides,

$$\int e^{-y}\,dy = \int dx$$

$$-e^{-y} = x + c$$

From the initial condition,

$$c = -1$$

Therefore, $\quad x = 1 - e^{-y}$

(3) $\dfrac{1}{x+2}\cdot\dfrac{dy}{dx}=-3y$ $\qquad\qquad (x=0,\ y=-1)$

[Sol] Since $x\neq-2$ and $y\neq0$, we can arrange the original equation as:

$$-\frac{dy}{3y}=(x+2)dx$$

Integrating both sides,

$$-\frac{1}{3}\int\frac{dy}{y}=\int(x+2)dx$$

$$-\frac{1}{3}\ln|y|=\frac{x^2}{2}+2x+c$$

$$y=\pm e^{-\frac{3}{2}x^2-6x}\cdot e^{-3c}$$

Replacing $\pm e^{-3c}$ with k,

$$y=ke^{-\frac{3}{2}x^2-6x}$$

From the initial condition,

$$k=-1$$

Therefore, $\qquad y=-e^{-\frac{3}{2}x^2-6x}$

(4) $e^{2x}\dfrac{dy}{dx}=e^{2y}$ $\qquad\qquad (x=-1,\ y=1)$

[Sol] Rearranging the original equation,

$$\frac{dy}{e^{2y}}=\frac{dx}{e^{2x}}$$

Integrating both sides,

$$\int\frac{dy}{e^{2y}}=\int\frac{dx}{e^{2x}}$$

$$-\frac{1}{2e^{2y}}=-\frac{1}{2e^{2x}}+c$$

$$\frac{1}{e^{2y}}=\frac{1}{e^{2x}}-2c$$

From the initial condition,

$$c=\frac{e^4-1}{2e^2}$$

Therefore, $\qquad \dfrac{1}{e^{2y}}=\dfrac{1}{e^{2x}}-\dfrac{e^4-1}{e^2}$

O 189 a

Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

Solve the following differential equations.

(1) $\dfrac{dy}{dx} = -3\dfrac{x}{y}$

[Sol] Rearranging the original equation,

$$y\,dy = -3x\,dx$$

Integrating both sides,

$$\int y\,dy = \int -3x\,dx$$

$$\frac{1}{2}y^2 = -\frac{3}{2}x^2 + c_1$$

$$y^2 = -3x^2 + 2c_1$$

Replacing $2c_1$ with c,

$$y^2 = -3x^2 + c$$

(2) $2y - \dfrac{dy}{dx} = x\dfrac{dy}{dx}$ …①

[Sol] (a) When $y = 0$, $\dfrac{dy}{dx} = 0$. This satisfies ①, so $y = 0$ is a solution.

(b) When $y \neq 0$, ① can be rewritten as:

$$\frac{dy}{2y} = \frac{dx}{x+1}$$

Integrating both sides,

$$\frac{1}{2}\int \frac{dy}{y} = \int \frac{dx}{x+1}$$

$$\frac{1}{2}\ln|y| = \ln|x+1| + c$$

$$\ln|y| = 2\ln|x+1| + 2c$$

$$y = \pm(x+1)^2 \cdot e^{2c}$$

Replacing $\pm e^{2c}$ with k,

$y = k(x+1)^2$ (where k is an arbitrary constant, $k \neq 0$)

However, when $k = 0$, $y = 0$. This satisfies (a).

Thus, the general solution is $y = k(x+1)^2$, where k is an arbitrary constant.

O 189 b

(3) $\dfrac{dy}{dx}=\dfrac{3y-1}{x+2}$ …①

[Sol] (a) When $y=\frac{1}{3}$, $\frac{dy}{dx}=0$. This satisfies ①, so $y=\frac{1}{3}$ is a solution.

(b) When $y\neq\frac{1}{3}$, ① can be rewritten as:

$$\frac{dy}{3y-1}=\frac{dx}{x+2}$$

Integrating both sides,

$$\int\frac{dy}{3y-1}=\int\frac{dx}{x+2}$$

$$\frac{1}{3}\ln|3y-1|=\ln|x+2|+c$$

$$\ln|3y-1|-3\ln|x+2|=3c$$

$$\frac{3y-1}{(x+2)^3}=\pm e^{3c}$$

Replacing $\pm e^{3c}$ with k,

$$3y-1=k(x+2)^3,\quad \text{(where } k \text{ is an arbitrary constant, } k\neq 0)$$

However, when $k=0$, $y=\frac{1}{3}$. This satisfies (a).

Thus, the general solution is $3y-1=k(x+2)^3$ where k is an arbitrary constant.

(4) $y\dfrac{dy}{dx}=\sqrt{1-4y^2}$

[Sol] Given $y\frac{dy}{dx}=\sqrt{1-4y^2}$ …①

(a) When $y=\pm\frac{1}{2}$, $\frac{dy}{dx}=0$. This satisfies ①, so $y=\pm\frac{1}{2}$ is a solution.

(b) When $y\neq\pm\frac{1}{2}$, ① can be rewritten as: $\dfrac{dy}{dx}=\dfrac{\sqrt{1-4y^2}}{y}$

Since $\sqrt{1-4y^2}\neq 0$, we can take the reciprocal of each side.

$$\frac{dx}{dy}=\frac{y}{\sqrt{1-4y^2}}$$

$$dx=\frac{y}{\sqrt{1-4y^2}}dy$$

Integrating both sides,

$$x=-\frac{1}{4}\sqrt{1-4y^2}+c$$

Therefore, the solutions are: $y=\pm\frac{1}{2}$ and $x=-\frac{1}{4}\sqrt{1-4y^2}+c$

O 190 a Differential Equations 2

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	1	2-

In each exercise, using the initial condition in the parentheses, solve each of the following differential equations.

(1) $\dfrac{dy}{dx} = 3y - 2$ $(x = 0,\ y = 2)$

[Sol] Since $y \neq \dfrac{2}{3}$, we can arrange the original equation as:

$$\frac{dy}{3y-2} = dx$$

Integrating both sides,

$$\int \frac{dy}{3y-2} = \int dx$$

$$\frac{1}{3}\ln|3y-2| = x + c$$

$$3y - 2 = \pm e^{3c} \cdot e^{3x}$$

Replacing $\pm e^{3c}$ with k, $3y - 2 = ke^{3x}$

From the initial condition, $k = 4$

Therefore, $3y - 2 = 4e^{3x}$

(2) $y + x\dfrac{dy}{dx} = 3$ $(x = 1,\ y = 1)$

[Sol] Since $x \neq 0$ and $y \neq 3$, we can arrange the original equation as:

$$\frac{dy}{3-y} = \frac{dx}{x}$$

Integrating both sides,

$$\int \frac{dy}{3-y} = \int \frac{dx}{x}$$

$$-\ln|3-y| = \ln|x| + c$$

$$y = 3 \pm \frac{1}{x} \cdot \frac{1}{e^c}$$

Replacing $\pm \dfrac{1}{e^c}$ with k, where $k \neq 0$, $y = 3 + \dfrac{k}{x}$

From the initial condition, $k = -2$

Therefore, $y = 3 - \dfrac{2}{x}$

O 190 b

(3) $e^{1-x}\dfrac{dy}{dx} = 2e^{3y}$ $\qquad (x = 1,\ y = 0)$

[Sol] Rearranging the original equation,

$$\frac{dy}{2e^{3y}} = \frac{dx}{e^{1-x}}$$

Integrating both sides,

$$\frac{1}{2}\int \frac{dy}{e^{3y}} = \int \frac{dx}{e^{1-x}}$$

$$\frac{1}{e^{3y}} = -6e^{x-1} - 6c$$

From the initial condition,

$$c = -\frac{7}{6}$$

Therefore, $\quad \dfrac{1}{e^{3y}} = -6e^{x-1} + 7$

(4) $\dfrac{2}{3x-1}\cdot\dfrac{dy}{dx} = y$ $\qquad (x = 1,\ y = e)$

[Sol] Rearranging the original equation,

$$\frac{dy}{y} = \frac{3x-1}{2}dx$$

Integrating both sides,

$$\int \frac{dy}{y} = \frac{1}{2}\int (3x-1)\,dx$$

$$\ln|y| = \frac{1}{2}\left(\frac{3x^2}{2} - x\right) + c$$

$$y = \pm e^{\frac{3}{4}x^2 - \frac{1}{2}x}\cdot e^{c}$$

Replacing $\pm e^{c}$ with k,

$$y = ke^{\frac{3}{4}x^2 - \frac{1}{2}x}$$

From the initial condition,

$$k = e^{\frac{3}{4}}$$

Therefore, $\quad y = e^{\frac{3}{4}x^2 - \frac{1}{2}x + \frac{3}{4}}$

O 191 a Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Determine the function $f(x)$ that satisfies each of the following equations.

(1) $f(x)=3\int_0^x f(t)\,dt+2$ …①

[Sol] Differentiating both sides of ① with respect to x,

$$f'(x)=\boxed{3f(x)}$$

Since $f(x)\neq 0$,

$$\frac{f'(x)}{f(x)}=\boxed{3} \quad \text{…②}$$

Integrating both sides of ② with respect to x,

$$\int\frac{f'(x)}{f(x)}dx=\int\boxed{3}\,dx$$

$$\ln|f(x)|=\boxed{3x}+c$$

$$\therefore f(x)=\pm\boxed{e^{3x}}\cdot e^{c}$$

Replacing $\pm e^{c}$ with k,

$$f(x)=k\cdot\boxed{e^{3x}}$$

From ①, $f(0)=\boxed{2}$. $\therefore k=\boxed{2}$ Thus, $f(x)=\boxed{2e^{3x}}$

(2) $f(x)=2\int_0^x tf(t)\,dt+2$ …①

[Sol] Differentiating both sides of ① with respect to x,

$$f'(x)=2xf(x)$$

Since $f(x)\neq 0$,

$$\frac{f'(x)}{f(x)}=2x \quad \text{…②}$$

Integrating both sides of ② with respect to x,

$$\int\frac{f'(x)}{f(x)}dx=\int 2x\,dx$$

$$f(x)=\pm e^{x^2}\cdot e^{c}$$

Replacing $\pm e^{c}$ with k,

$$f(x)=ke^{x^2}$$

From ①, $f(0)=2$. $\therefore k=2$ Thus, $f(x)=2e^{x^2}$

O 191 b

2. Determine the function $f(x)$ that satisfies the equation
$$\int_1^x (4t+5)f(t)\,dt = 3(x+2)\int_1^x f(t)\,dt, \quad \text{where } f(0)=1.$$

(**Hint:** Differentiate both sides of the given equation twice.)

[Sol] Given $\displaystyle\int_1^x (4t+5)f(t)\,dt = 3(x+2)\int_1^x f(t)\,dt,$ $\qquad\cdots①$

Differentiating both sides of ① with respect to x,

$$(4x+5)f(x) = 3\int_1^x f(t)\,dt + 3(x+2)f(x)$$

$$\therefore (x-1)f(x) = 3\int_1^x f(t)\,dt \qquad \cdots②$$

Differentiating both sides of ② with respect to x,

$$f(x) + (x-1)f'(x) = 3f(x)$$

$$\therefore (x-1)f'(x) = 2f(x)$$

Since $f(x) \neq 0$,

$$\frac{f'(x)}{f(x)} = \frac{2}{x-1} \qquad \cdots③$$

Integrating both sides of ③ with respect to x,

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{2}{x-1}\,dx$$

$$\ln|f(x)| = 2\ln|x-1| + c$$

$$f(x) = \pm e^c (x-1)^2$$

Replacing $\pm e^c$ with k,

$$f(x) = k(x-1)^2$$

Since $f(0)=1$, $\quad k=1$

Therefore, $f(x) = (x-1)^2$

KUMON

O 192 a Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	1	-	2-

1. Given the functions $f(x)$, $g(x)$ and $F(x)$, where $F(x)=\dfrac{f(x)}{g(x)}$, ...①

complete the following exercises.

(1) Given that $f'(x)=g(x)$ and $g'(x)=f(x)$, determine the differential equation that $F(x)$ satisfies.

(**Hint:** Use the Quotient Rule.)

[Sol] Differentiating ①,

$$F'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

From the given,

$$F'(x)=\frac{[g(x)]^2-[f(x)]^2}{[g(x)]^2}=\frac{[g(x)]^2}{[g(x)]^2}-\frac{[f(x)]^2}{[g(x)]^2}=1-[F(x)]^2$$

When $y=F(x)$,

it satisfies the differential equation: $\dfrac{dy}{dx}=1-\boxed{y^2}$ ...②

(2) Given that $f(0)=0$ and $g(x)\neq 0$, determine $F(x)$.

[Sol] From ①, when $y=F(x)=\pm 1$, $f(x)=\pm g(x)$

Therefore, if $f(0)=0$, then $g(0)=\boxed{0}$.

Since $g(x)\neq 0$, $y\neq\boxed{\pm 1}$

Rewriting ②, $\boxed{\dfrac{1}{1-y^2}}\cdot dy=dx$ ...③

Integrating both sides of ③,

$$\int\frac{dy}{1-y^2}=\int dx$$

$$\frac{1}{2}\int\left(\frac{1}{1-y}+\frac{1}{1+y}\right)dy=\int dx$$

$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|=x+c$$

$$\frac{1+y}{1-y}=ke^{2x}\quad(\text{where } k=\pm e^{2c})$$

Therefore, the differential equation is: $\dfrac{1+F(x)}{1-F(x)}=ke^{2x}$ ...④

Since $f(0)=0$ and $g(x)\neq 0$, $F(0)=0$ and $k=1$ (from ④)

Thus, $\dfrac{1+F(x)}{1-F(x)}=e^{2x}$ $\therefore F(x)=\dfrac{e^{2x}-1}{e^{2x}+1}$

2. Given $f(x)$ and $g(x)$ which satisfy the following equations,

$$\begin{cases} f'(x) + 2g'(x) - f(x) - 2g(x) = 0 & \cdots ① \\ 2f'(x) - g'(x) + x[2f(x) - g(x)] = 0 & \cdots ② \end{cases}$$

complete the following exercises.

(1) Find $f(x) + 2g(x)$.

[Sol] Letting $y = f(x) + 2g(x)$,

From ①, $\dfrac{dy}{dx} - \boxed{y} = 0$ …③

Rearranging ③, $\dfrac{dy}{\boxed{y}} = dx$

Integrating both sides, $y = \pm e^{c} \cdot e^{x}$

Replacing $\pm e^{c}$ with k_1, $y = f(x) + 2g(x) = k_1 e^{x}$ …④

(2) Find $2f(x) - g(x)$.

[Sol] Letting $y = 2f(x) - g(x)$,

From ②, $\dfrac{dy}{dx} + xy = 0$ …⑤

Rearranging ⑤, $\dfrac{dy}{y} = -x\,dx$

Integrating both sides, $y = \pm e^{c} \cdot e^{-\frac{1}{2}x^2}$

Replacing $\pm e^{c}$ with k_2, $y = 2f(x) - g(x) = k_2 e^{-\frac{1}{2}x^2}$ …⑥

(3) When $f(0) = 5$ and $g(0) = 0$, find $f(x)$ and $g(x)$.

[Sol] $$\begin{cases} f(x) + 2g(x) = k_1 e^{x} & \cdots ④ \\ 2f(x) - g(x) = k_2 e^{-\frac{1}{2}x^2} & \cdots ⑥ \end{cases}$$

Solving for $f(x)$ and $g(x)$,

$$f(x) = \frac{1}{5}\left(k_1 e^{x} + 2k_2 e^{-\frac{1}{2}x^2}\right), \quad g(x) = \frac{1}{5}\left(2k_1 e^{x} - k_2 e^{-\frac{1}{2}x^2}\right)$$

Since $f(0) = 5$ and $g(0) = 0$, $k_1 = 5,\ k_2 = 10$

Therefore, $f(x) = e^{x} + 4e^{-\frac{1}{2}x^2}$ and $g(x) = 2e^{x} - 2e^{-\frac{1}{2}x^2}$

Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Let $f(x)$ be a differentiable function for all x such that $f(x+y)=f(x)\cdot f(y)$ is always true. Determine $f(x)$.

[Sol] Given $\quad f(x+y)=f(x)\cdot f(y) \quad \cdots①$

Differentiating ① with respect to y,

$$f'(x+y)=f(x)\cdot f'(y)$$

Letting $y=0$, $\quad f'(y)=A \quad$ (where A is a constant)

$$\therefore f'(x)=f'(x+0)=\boxed{Af(x)} \quad \cdots②$$

(a) When $f(x)\neq 0$,

$$\frac{f'(x)}{f(x)}=\boxed{A}$$

Integrating both sides with respect to x,

$$\int\frac{f'(x)}{f(x)}dx=\int\boxed{A}dx$$

$$\therefore \ln\left|\boxed{f(x)}\right|=\boxed{Ax}+c$$

$$\therefore f(x)=\pm e^{\boxed{Ax+c}}=\pm e^{c}(e^{A})^{x}$$

Letting $k=\pm e^{c}$, and $a=e^{A}$ (where $a>0$),

$$f(x)=\boxed{ka^{x}} \quad \cdots③$$

(b) When $f(x)=0$, both ② and ③ are satisfied, when $k=0$.

Thus, $f(x)=\boxed{ka^{x}}$ (where $a>0$) $\quad \cdots④$

Substituting $\quad x=y=0$ into ①,

$$f(0)=[f(0)]^2$$

Letting $\quad f(0)=X,\quad X=X^2$

Therefore, $f(0)=0$ or $\boxed{1}$

When $f(0)=0$, $\quad$ ④ gives $k=0$ $\quad \therefore f(x)=\boxed{0}$

When $f(0)=\boxed{1}$, $\quad$ ③ gives $k=1$ $\quad \therefore f(x)=\boxed{a^{x}}$

Thus, the functions are: $f(x)=\boxed{0}$ and $f(x)=\boxed{a^{x}}$ (where $a>0$)

O 193 b

2. In each exercise, determine the differentiable function $f(x)$ which satisfies the given equation.

(1) $f(xy) = f(x) \cdot f(y)$

[Sol] Given $\quad f(xy) = f(x) \cdot f(y) \qquad \cdots ①$

Differentiating both sides of ① with respect to y,

$$xf'(xy) = f(x) \cdot f'(y)$$

Letting $y = 1$,

$$xf'(xy) = xf'(x) = Af(x) \quad \text{(where } A \text{ is a constant)} \qquad \cdots ②$$

(a) When $f(x) \neq 0$,

$$\frac{f'(x)}{f(x)} = \frac{A}{x}$$

Integrating both sides with respect to x,

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{A}{x} dx$$

$$\ln|f(x)| = A\ln|x| + c$$

$$\therefore f(x) = kx^A \qquad \text{(where } k = \pm e^c\text{)} \qquad \cdots ③$$

(b) When $f(x) = 0$, both ② and ③ are satisfied.

Therefore, $f(x) = kx^A$

Replacing $x = y = 1$ into ①, $\quad f(1) = 0$ or 1

Thus, the functions are: $\quad f(x) = 0$ and $f(x) = x^A$

(2) $f(x+y) = f(x) + f(y)$ $\qquad$ (**Hint:** Let $f'(x) = A$)

[Sol] Given $\quad f(x+y) = f(x) + f(y) \qquad \cdots ①$

Differentiating both sides of ① with respect to x,

$$f'(x+y) = f'(x)$$

Since this is true for all values of y,

Letting $\quad f'(x) = A$,

$$f(x) = \int f'(x) dx$$

$$= Ax + c \qquad \text{(where } A \text{ is a constant)}$$

From ①, when $x = 0$, $f(0) = 0$.

$$\therefore c = 0$$

Therefore, $f(x) = Ax$

O 194 a Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx} = x + y$ $\quad\cdots$①

[Sol] Letting $x + y = u$ and differentiating both sides with respect to x,

$$1 + \frac{dy}{dx} = \boxed{\frac{du}{dx}}$$

$$\therefore \frac{dy}{dx} = \boxed{\frac{du}{dx} - 1}$$

Substituting this value into ①,

$$\boxed{\frac{du}{dx} - 1} = x + y$$

$$= u$$

$$\therefore \frac{du}{dx} = \boxed{u + 1} \quad \cdots②$$

(a) When $u = -1$, it satisfies ②, so $u = -1$ is a solution.

(b) When $u \neq -1$, ② can be rewritten as: $\dfrac{du}{u+1} = dx$

Integrating both sides,

$$\int \frac{du}{u+1} = \int dx$$

$$\ln|u + 1| = x + c$$

$$\therefore u + 1 = \pm e^{x+c}$$

Replacing $\pm e^c$ with k, $\quad u = ke^x - 1 \quad$ (where $k \neq 0$)

However, when $k = 0$, $u = -1$. This satisfies (a).

Since $x + y = u$,

$$x + y = ke^x - 1$$

$$[y = ke^x - x - 1]$$

O 194 b

(2) $(x+y)\dfrac{dy}{dx} = 2(x+y)+1$ $\qquad\cdots①$

[Sol] Letting $x+y=u$.

Differentiating both sides with respect to x,

$$1+\frac{dy}{dx}=\frac{du}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{du}{dx}-1$$

Substituting this value into ①,

$$u\left(\frac{du}{dx}-1\right)=2u+1$$

$$u\frac{du}{dx}=3u+1 \qquad\cdots②$$

(a) When $u=-\dfrac{1}{3}$, it satisfies ②, so $u=-\dfrac{1}{3}$ is a solution.

(b) When $u\neq-\dfrac{1}{3}$, ② can be rewritten as: $\dfrac{u}{3u+1}du=dx$

Integrating both sides,

$$\int\frac{u}{3u+1}du=\int dx$$

$$\frac{1}{3}\int\left(1-\frac{1}{3u+1}\right)du=x+c$$

$$\frac{1}{3}\left(u-\frac{1}{3}\ln|3u+1|\right)=x+c$$

$$u-\frac{1}{3}\ln|3u+1|=3x+3c$$

Since $x+y=u$,

$$x+y-\frac{1}{3}\ln|3(x+y)+1|=3x+3c$$

$$\therefore \ln|3(x+y)+1|=3y-6x-9c$$

$$\therefore 3(x+y)+1=\pm e^{3y-6x-9c}$$

Replacing $\pm e^{-9c}$ with k,

$$3x+3y+1=ke^{3y-6x} \qquad (\text{where } k\neq 0)$$

However, when $k=0$, $u=x+y=-\dfrac{1}{3}$. This satisfies (a).

Therefore, the general solution is: $3x+3y+1=ke^{3y-6x}$

KUMON

O 195 a

Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx}=2x-y-1$ $\cdots①$

(**Hint:** Let $u=2x-y-1$, and then differentiate u with respect to x.)

[Sol] Letting $u=2x-y-1$,

$$\frac{du}{dx}=2-\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=2-\frac{du}{dx}$$

Substituting this value into ①,

$$2-\frac{du}{dx}=u$$

$$\therefore \frac{du}{dx}=2-u \qquad \cdots②$$

(a) When $u=2$, it satisfies ②, so $u=2$ is a solution.

(b) When $u\neq 2$, ② can be rewritten as: $\dfrac{du}{2-u}=dx$

Integrating both sides,

$$\int\frac{du}{2-u}=\int dx$$

$$-\ln|2-u|=x+c$$

$$\therefore 2-u=\pm e^{-x-c}$$

Since $u=2x-y-1$,

$$2-(2x-y-1)=\pm e^{-x-c}$$

Replacing $\pm e^{-c}$ with k,

$$2-(2x-y-1)=ke^{-x} \qquad \text{(where } k\neq 0)$$

However, when $k=0$, $u=2x-y-1=2$. This satisfies (a).

Thus, the general solution is: $2-(2x-y-1)=ke^{-x}$

$[y=ke^{-x}+2x-3]$

O 195 b

(2) $\dfrac{dy}{dx}=\dfrac{1-x-y}{x+y}$ $\quad\cdots①$

(**Hint:** Let $u=x+y$, and then differentiate u with respect to x.)

[Sol] Letting $u=x+y$,

$$\frac{du}{dx}=1+\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{du}{dx}-1$$

Substituting this value into ①,

$$\frac{du}{dx}-1=\frac{1-u}{u}$$

$$\therefore \frac{du}{dx}=\frac{1}{u} \qquad \cdots②$$

Rewriting ②,

$$udu=dx$$

Integrating both sides,

$$\int udu=\int dx$$

$$\frac{1}{2}u^2=x+c_1$$

Since $u=x+y$,

$$\frac{1}{2}(x+y)^2=x+c_1$$

$$(x+y)^2=2x+2c_1$$

Replacing $2c_1$ with c,

$$(x+y)^2=2x+c \qquad \text{(where } c \text{ is an arbitrary constant)}$$

Thus, the general solution is: $(x+y)^2=2x+c$

Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%-
(mistakes) 0	-	-	-	-

Solve each of the following differential equations.

(1) $\dfrac{dy}{dx}=\dfrac{1}{xy^2(3+x^3)}$ …①

[Sol] Rewriting ①,

$$y^2dy=\frac{dx}{x(3+x^3)}$$

$$\therefore\ y^2dy=\frac{x^2}{x^3(3+x^3)}dx \quad \cdots②$$

Letting $u=3+x^3$,

$$\frac{du}{dx}=3x^2$$

Substituting this value into ②,

$$y^2dy=\frac{1}{3}\cdot\frac{du}{(u-3)u}$$

Integrating both sides,

$$\int y^2dy=\frac{1}{3}\int\frac{du}{(u-3)u}$$

$$\frac{1}{3}y^3=\frac{1}{9}\int\left(\frac{1}{u-3}-\frac{1}{u}\right)du$$

$$\frac{1}{3}y^3=\frac{1}{9}\ln\left|\frac{u-3}{u}\right|+c_1$$

$$y^3=\frac{1}{3}\ln\left|\frac{u-3}{u}\right|+3c_1$$

Replacing $3c_1$ with c,

$$y^3=\frac{1}{3}\ln\left|\frac{u-3}{u}\right|+c \quad \text{(where } c \text{ is an arbitrary constant)}$$

Therefore, the general solution is:

$$y^3=\frac{1}{3}\ln\left|\frac{x^3}{3+x^3}\right|+c$$

O 196 b

(2) $x\left(\dfrac{dy}{dx}+1\right)+\tan(x+y)=0 \qquad \cdots①$

(**Hint:** Replace c with $\ln k_1$, where c and k_1 are constants.)

[Sol] Letting $u=x+y$,

$$\frac{du}{dx}=1+\frac{dy}{dx}$$

Substituting this value into ①,

$$x\frac{du}{dx}+\tan u=0 \qquad \cdots②$$

(a) When $u=n\pi$ (where n is an integer), it satisfies ②.

$\therefore u=n\pi$ is a solution.

(b) When $u\neq n\pi$ (where n is an integer), ② can be rewritten as:

$$\frac{du}{\tan u}=-\frac{dx}{x}$$

Integrating both sides,

$$\int\frac{du}{\tan u}=-\int\frac{dx}{x}$$

$$\int\frac{\cos u}{\sin u}du=-\int\frac{dx}{x}$$

$$\therefore \ln|\sin u|=-\ln|x|+c$$

Replacing c with $\ln k_1$,

$$\ln|\sin u|=-\ln|x|+\ln k_1 \qquad (\text{where } k_1>0)$$

$$|x|\cdot|\sin u|=k_1$$

$$\therefore x\sin u=\pm k_1$$

However, when $k_1=0$, $u=x+y=n\pi$. This satisfies (a).

Replacing $\pm k_1$ with k, $\quad x\sin u=k$

Since $u=x+y$,

The general solution is: $\quad x\sin(x+y)=k$

O 197 a Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

In each exercise, using the initial condition in the parentheses, solve the given differential equation.

(1) $2xy\dfrac{dy}{dx} = x^2 + y^2$ $(x = -1,\ y = 0)$

[Sol] Since $x \neq 0$,

Rearranging the original equation,

$$2\frac{y}{x}\cdot\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2 \quad \cdots①$$

Letting $u = \dfrac{y}{x}$, $\cdots②$

$$y = ux \qquad \therefore\ \frac{dy}{dx} = \frac{du}{dx}x + u \quad \cdots③$$

Substituting ② and ③ into ①,

$$2u\left(\frac{du}{dx}x + u\right) = 1 + u^2$$

$$2ux\frac{du}{dx} = 1 - u^2$$

$$\therefore\ \frac{2u}{1-u^2}du = \frac{dx}{x}$$

Integrating both sides,

$$\int\frac{2u}{1-u^2}du = \int\frac{dx}{x}$$

$$-\ln|1-u^2| = \ln|x| + c$$

$$\ln|(1-u^2)x| = -c$$

$$(1-u^2)x = \pm e^{-c}$$

Replacing $\pm e^{-c}$ with k,

$$(1-u^2)x = k \quad (\text{where } k \neq 0)$$

Since $u = \dfrac{y}{x}$,

$$x^2 - y^2 = kx$$

Using the initial condition, $k = -1$

Therefore, the general solution is: $x^2 - y^2 + x = 0$

O 197 b

(2) $x\dfrac{dy}{dx}-(y-x)=0$ $\qquad (x=1,\ y=1)$

[Sol] Since $x \neq 0$,

Rearranging the original equation,

$$\frac{dy}{dx}=\frac{y}{x}-1 \qquad \cdots①$$

Letting $u=\dfrac{y}{x}$, $\qquad \cdots②$

$$y=ux \qquad \therefore \frac{dy}{dx}=\frac{du}{dx}x+u \qquad \cdots③$$

Substituting ② and ③ into ①,

$$\frac{du}{dx}x+u=u-1$$

$$du=-\frac{dx}{x}$$

Integrating both sides,

$$\int du=-\int\frac{dx}{x}$$

$$u=-\ln|x|+c$$

Since $u=\dfrac{y}{x}$,

$$\frac{y}{x}=-\ln|x|+c$$

$$y=x(c-\ln|x|)$$

Using the initial condition, $c=1$

Therefore, the general solution is: $\quad y=x(1-\ln|x|)$

KUMON

O 198 a Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	–

1. Let $f(x)$ be a 3rd degree polynomial such that $f(x)$ is divisible by $f'(x)$, $f(1)=\frac{8}{3}$ and $f(-1)=0$. Determine $f(x)$.

[Sol] Since $f'(x)$ is one degree less than $f(x)$,

Let $f(x)=(ax+b)f'(x)$.

Letting $y=f(x)$,

$$y=(ax+b)\frac{dy}{dx} \quad \cdots ①$$

Since $f(1)=\frac{8}{3}\neq 0$, $\quad y\neq 0$

Rewriting ①,

$$\frac{dy}{y}=\frac{dx}{ax+b}$$

Integrating both sides,

$$\int\frac{dy}{y}=\int\frac{dx}{ax+b}$$

$$\ln|y|=\frac{1}{a}\ln|ax+b|+c$$

$$y=\pm e^{c}(ax+b)^{\frac{1}{a}}$$

Replacing $\pm e^{c}$ with k,

$$y=k(ax+b)^{\frac{1}{a}}$$

Since $y=f(x)$ is a 3rd degree polynomial,

$$\frac{1}{a}=3 \quad \therefore a=\frac{1}{3}$$

Therefore, $f(x)=k\left(\frac{1}{3}x+b\right)^3 \quad \cdots ②$

Substituting the original condition into ②,

$$\frac{8}{3}=k\left(\frac{1}{3}+b\right)^3, \quad 0=k\left(-\frac{1}{3}+b\right)^3$$

Since $k\neq 0$, $\quad k=9,\ b=\frac{1}{3}$

Thus, $f(x)=\frac{1}{3}(x+1)^3$

2. Given that y is a function of x satisfying the differential equation $y'' - y = 2\sin x$, complete the following exercises.

(1) If $y = e^x u - \sin x$ (where u is a function of x), determine the differential equation that u satisfies.

[Sol] $y = e^x u - \sin x$ …①

$y' = e^x u + e^x u' - \cos x$ …②

$y'' = e^x u'' + 2e^x u' + e^x u + \sin x$ …③

Substituting ① and ③ into the given differential equation,

$$(e^x u'' + 2e^x u' + e^x u + \sin x) - (e^x u - \sin x) = 2\sin x$$

$$e^x u'' + 2e^x u' = 0$$

$$\therefore u'' + 2u' = 0$$

(2) Given $u(0) = 3$ and $u'(0) = -2$, determine y.

[Sol] From (1), $u'' + 2u' = 0$

Letting $u' = t$,

$$t' + 2t = 0 \quad \text{…①}$$

Rewriting ①, $\dfrac{dt}{dx} = -2t$ …②

(a) When $t = 0$, it satisfies ②, so $t = 0$ is a solution.

(b) When $t \neq 0$, ② can be rewritten as: $\dfrac{dt}{t} = -2dx$

Integrating both sides,

$$\int \frac{dt}{t} = -\int 2dx$$

$$\ln|t| = -2x + c$$

$$t = \pm e^{-2x+c}$$

Replacing $\pm e^c$ with k,

$$t = ke^{-2x} \quad \text{(where } k \neq 0\text{)} \quad \text{…③}$$

Since $u' = t$,

$$u = -\frac{k}{2}e^{-2x} + c \quad \text{…④}$$

Substituting the original conditions into ③ and ④,

$$k = -2, c = 2$$

Therefore,

$$u = e^{-2x} + 2$$

$$y = e^x(e^{-2x} + 2) - \sin x$$

$$= 2e^x + e^{-x} - \sin x$$

O 199 a Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%–
(mistakes) 0	–	–	–	1–

1. An object dropped in the air encounters air resistance proportional to its velocity. Assume that the acceleration due to gravity is g. Express the velocity v as a function of t, where t represents the number of seconds after the object begins to fall.

(Hint: At time $t=0$, the velocity is 0.)

[Sol] Given a body with mass m, the downward force by gravity is mg, and the upward force by air resistance is Bv (where B is a constant). Letting the downward direction be positive,

$$m\frac{dv}{dt} = mg - Bv$$

Since $m > 0$ and $mg - Bv > 0$, we can rewrite the equation as:

$$\frac{dv}{\boxed{g - \frac{B}{m}v}} = dt$$

Integrating both sides,

$$\int \frac{dv}{g - \frac{B}{m}v} = \int dt$$

$$-\frac{m}{B}\ln\left|g - \frac{B}{m}v\right| = t + c \qquad \cdots ①$$

When $t=0$, $v=0$.

$$\therefore c = -\frac{m}{B}\ln g \qquad \cdots ②$$

Substituting ② into ①,

$$-\frac{m}{B}\ln\left|g - \frac{B}{m}v\right| = t - \frac{m}{B}\ln g$$

$$\therefore v = \frac{mg}{B}\left(1 - e^{-\frac{B}{m}t}\right)$$

2. Radioactive substances decay over time and lose mass at a rate which is proportional to the amount of mass remaining. If only $\frac{1}{3}$ of the original mass remains after 1600 years, how much will be left after another 800 years?

[Sol] At time t, let x be the fraction of the initial mass remaining. $(x>0)$

Letting $-\frac{dx}{dt}$ be the rate of decay,

$$\frac{dx}{dt}=-Bx$$

(where B is the constant of proportionality, and $B>0$)

Rearranging,

$$\frac{dx}{x}=-Bdt$$

Integrating both sides,

$$\int\frac{dx}{x}=-B\int dt$$

$$x=\pm e^{-Bt}\cdot e^{C}$$

Replacing $\pm e^{C}$ with k,

$$x=ke^{-Bt} \qquad \cdots①$$

Since $x=1$ when $t=0$,

$$k=1$$

Substituting this value into ①,

$$x=e^{-Bt} \qquad \cdots②$$

Since $x=\frac{1}{3}$ when $t=1600$,

$$\frac{1}{3}=e^{-1600B} \qquad \cdots③$$

Solving ② when $t=2400$,

$$x=e^{-2400B}=(e^{-1600B})^{\frac{3}{2}}$$

$$=\left(\frac{1}{3}\right)^{\frac{3}{2}}=\frac{\sqrt{3}}{9}$$

Therefore, after 2400 years, $\frac{\sqrt{3}}{9}$ of the original mass will remain.

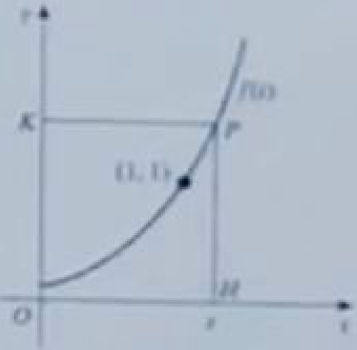
Differential Equations 3

Time : to : Date Name

100%	90%	80%	70%	69%-

1. Let $f(x)$ be a differentiable function with $x>0$, $y>0$ and $f(1)=1$. Let $OHPK$ be a rectangle formed by the points $O(0,0)$, $H(t,0)$, $P(t, f(t))$ and $K(0, f(t))$.

Determine the function $y=f(x)$ such that the area bounded by $x=0$, $x=t$, $y=0$ and $y=f(x)$ is always $\frac{1}{5}$ the area of rectangle $OHPK$ (where $t>0$).



[Sol] The area of rectangle $OHPK$ is: $t \cdot f(t)$

From the given,

$$\int_0^t f(x)dx = \frac{1}{5}t \cdot f(t) \quad \cdots ①$$

Differentiating both sides of ① with respect to t,

$$f(t) = \frac{1}{5}f(t) + \frac{1}{5}t \cdot f'(t)$$

$$\therefore 4f(t) = t \cdot f'(t)$$

When $t>0$, $f(t)>0$

$$\therefore \frac{f'(t)}{f(t)} = \frac{4}{t}$$

$$\therefore \int \frac{f'(t)}{f(t)}dt = \int \frac{4}{t}dt$$

$$\ln f(t) = 4\ln t + c$$

$$f(t) = e^c t^4$$

Replacing e^c with k,

$$f(t) = kt^4$$

Since $f(1)=1$, $k=1$

Thus, $f(x) = x^4$

O 200 b

2. Determine the differentiable function $f(x)$ which satisfies the following equation.

$$\int_0^x f(t)\,dt = x^3 - 3x^2 + x + \int_0^x (x-t)f(t)\,dt$$

[Sol] Rewriting the given, $\int_0^x f(t)\,dt = x^3 - 3x^2 + x + x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt$ …①

Differentiating ① with respect to x,

$$f(x) = 3x^2 - 6x + 1 + \int_0^x f(t)\,dt + xf(x) - xf(x)$$

$$\therefore f(x) = 3x^2 - 6x + 1 + \int_0^x f(t)\,dt \qquad \text{…②}$$

Differentiating ② with respect to x,

$$f'(x) = 6x - 6 + f(x)$$

Letting $y = f(x)$,

$$\frac{dy}{dx} = 6x - 6 + y \qquad \text{…③}$$

Letting $u = 6x - 6 + y$,

$$\frac{du}{dx} = 6 + \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} - 6$$

Substituting this value into ③,

$$\frac{du}{dx} - 6 = u \qquad \text{…④}$$

Rearranging ④,

$$\frac{du}{u+6} = dx$$

Integrating both sides,

$$\int \frac{du}{u+6} = \int dx$$

$$\therefore \ln|u+6| = x + c \qquad \text{…⑤}$$

From ②, when $x = 0$, $y = 1$. $\quad \therefore u = -5$

Substituting into ⑤, $\quad c = 0$

$$\ln|u+6| = x$$

$$\therefore u + 6 = \pm e^x$$

Since $u = 6x - 6 + y$,

$$(6x - 6 + y) + 6 = \pm e^x$$

$$\therefore y = \pm e^x - 6x$$

However, $y = -e^x - 6x$ does not satisfy ②.

Thus, $f(x) = e^x - 6x$